Totally asymmetric coalescence process

L. T. Costa* and A. D. Ramos†

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Abstract

Mathematical modeling used in describing coalescence processes is extensively documented in the literature, but not all of these theoretical frameworks encompass the concept of coalescence itself. This research presents the Totally Asymmetric Coalescence Process (TACP), a new class of interacting particles in which the environment functions in the interaction process. Theoretical results and numerical analysis are presented for the TACP.

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1 Introduction

In the interacting particle systems theory, the assumption that the set of particles does not alter in the process of interaction is a well-established concept. This assumption, known as the constant-length assumption, is not the only possible one. A. Toom and V. Malyshev [1, 2] have suggested an alternative method, referred to in the literature as *variable or complex architecture*[3], process with *variable length*[4], and *random substitution*[5]. Unlike traditional procedures, the space changes during the evolution process in this approach. In this research, we will adopt the terminology of interacting particle systems with variable length [6, 7, 8, 2, 9].

This notion of variable-length, in finite systems, has been employed in biology, in the context-free L-systems to describe the development of higher plants and complex branching structures [10]. It can also be seen in some models proposed to determine the nucleotide sequence evolution and patterns in a genome [11]. In the context of genomic evolution modeling, presuming its particles are in $\mathbb{N}$, the so-called *expansion-modification system* [12, 13, 14]. is observed. In addition, variable-length has been employed in the study of the robustness of quasiperiodic structures [15] associated with random perturbations of Fibonacci sequences and Penrose tilings. Particularly, the existence of a positive topological entropy was determined. It was later confirmed [16] that this topological feature holds for a vast class.

*Rural Federal University of Pernambuco, CODAI-UFRPE, Sourena Mata/PE, 54735-000, Brazil; e-mail: `leon.tarquino@ufrpe.br`

†the Federal University of Pernambuco, Department of Statistics, Recife/PE, 50740-540, Brazil; e-mail: `alex@de.ufpe.br`

A more general formalization for this process has been introduced by A. Toom and V. Malyshev, both of whom made great contributions in providing valuable insights into this study's scope and demonstrating the continued advancement and innovation in the field. They had diverse motivations. V. Malyshev [1, 17] was inspired by computer science and its associated with quantum computation glitches and some concerns about modern physics, particularly quantum gravity. Contrarily, A. Toom [9] was inspired by the *positive rates conjecture*, which can be informally given as follows: *all one - dimensional particle systems with non-degenerate local interaction are ergodic, that is cannot display analogs of phase transitions*. This conjecture was rebutted by Peter Gács [18].

We believe that the particle process with variable length more accurately elucidates the concept of coalescence existing in some real-world phenomena. This will certainly aid in the development of this theory and unlock new possibilities for its application.

Informally, in our process, the particles will be located at $\mathbb{Z}$. We presume that the environment will only trigger a positive or negative stimulus. Therefore, our particles take on only two values, referred to here as *plus* and *minus*, represented by $\oplus$ and $\ominus$ respectively. The process we have evaluated here is called the Asymmetric Coalescence Process (TACP) and has discrete time. At each time step, two operations occur. The first operation called *flip,* converts every minus into a plus with probability β, independently of what occurs at other locations. We can imagine the flip as a change from negative to positive stimulus for the particle. The second operation, called *coalescence*, functions as follows: if a particle with a positive stimulus is a left neighbor of a particle with a negative stimulus, the particle with a positive stimulus will be attracted to its left neighbor, and with probability, α, this attraction will be powerful enough to cause a merge between the two particles into one (coalesce). In the new particle developed, the negative stimulus will be dominant and will be independent of what occurs in other locations.

Beginning from the delta measure concentrated on the configuration where all components are in the negative state $\ominus$, we can validate the existence of a phase transition between ergodic and non-ergodic behavior in the TACP in the parameter space. In the non-ergodic regime, we show the existence of at least two linearly independent invariant measures. The density of positive stimuli $\oplus$ is one, while in the other it is less than one in one of these measures. The TACP displays a first-order phase transition, which is unexpected given the analogy with some traditional processes (such as percolation processes) with constant-length.

The structure of this research is as follows: In Section 2, we highlight our major outcomes; a numerical analysis with mean-field approximation and Monte Carlo simulation is illustrated in Section 3; In Section 4, we present our operators; Section 5 focuses on the proofs of Theorems 2, 3, and 4'. Owing to the large number of theoretical tools needed to prove Theorem 1, we dedicate Sections 6 and 7 to this task. Finally, some open questions are introduced in Section 8.

2 Main Results

We refer to a non-empty finite set $\mathcal{A}$ as an *alphabet* and its elements as *letters*. A *word* in the alphabet $\mathcal{A}$ is described as a finite sequence of elements of $\mathcal{A}$. The length of a word W is equal to the number of its letters and is represented by $|W|$. The empty word, depicted by Λ, has a length of zero. The set of all words in the alphabet $\mathcal{A}$ is called the *dictionary* and is signified by $dic(\mathcal{A})$. The set of integers is denoted by $\mathbb{Z}$ and the set of all bi-infinite sequences comprising of elements from $\mathcal{A}$ is represented by $\mathcal{A}^{\mathbb{Z}}$.

Let $\mathcal{A}$ be a set and represent by $\mathbb{A}$ the discrete topology on $\mathcal{A}$. We contemplate probability measures on the σ-algebra $\mathbb{A}^{\mathbb{Z}}$ on the product space $\mathcal{A}^{\mathbb{Z}}$ endowed with the product of discrete topologies on all the copies of $\mathcal{A}$.

In the usual way, we define the translation on $\mathbb{Z}$, then on $\mathcal{A}^{\mathbb{Z}}$, and finally on the set of probability measures on $\mathcal{A}^{\mathbb{Z}}$. A measure is said to be *translation-invariant* if it is invariant under all translations, that is,

$$\mu(a_1, \ldots, a_n) = \mu(s_{i+1} = a_1, \ldots, s_{i+n} = a_n), \text{ for all } i \in \mathbb{Z}$$

For any word $W \in dic(\mathcal{A})$ (including the empty word), we have:

$$\mu(W) = \sum_{a \in \mathcal{A}} \mu(W, a) = \sum_{a \in \mathcal{A}} \mu(a, W).$$

With this data, we can finally conclude that $\mathcal{M}$ is the set of all translation-invariant probability measures on $\mathcal{A}^{\mathbb{Z}}$ that are described for any word $W \in dic(\mathcal{A})$, such as the empty word, and any letter $a \in \mathcal{A}$. The concatenation of the word W and letter a can happen in either order, i.e., (W, a) or (a, W).

In this context, $\mu\mathsf{P}^t$ refers to the t-th composition of the measure μ and the operator P. A measure μ is called invariant for the operator P if $\mu\mathsf{P} = \mu$, meaning that the measure remains unchanged after one application of P.

If the limit $\lim\limits_{t\to\infty} \mu\mathsf{P}^t$ exists and is the same for any initial measure μ, then the process is called ergodic. This means that as the number of iterations of the operator P grows larger, the result converges to a unique and invariant distribution, regardless of the initial measure.

The motivation behind this research comes from the concept of coalescence and a new class of interacting particle systems with variable length presented in [2]. In this work, we suggest and research a discrete time procedure in which the particles are placed on the integer line $\mathbb{Z}$ and can only predict one of two stimuli: positive (denoted by $\oplus$) or negative (denoted by $\ominus$). The procedure operates on $\mathcal{M}_{\{\ominus,\oplus\}}$, the set of translation-invariant probability measures on $\{\ominus, \oplus\}^{\mathbb{Z}}$.

For simplicity, we refer to this process as the Totally Asymmetric Coalescence Process (TACP).

The first operator, known as the *flip* operator, is denoted by F_β. This operator is of frequent length and changes negative stimuli into positive stimuli with probability β, independently of any other happenings.

The second operator, known as the *coalesce* operator, is denoted by C_α and maps $\mathcal{M}_{\ominus,\oplus}$ to itself. Below the action of this variable-length operator, if a particle with a positive stimulus is the left neighbor of a particle with a negative stimulus, they will be criped to each other with probability α. The attraction is so strong that these two particles merge into one, developing a single particle with a negative stimulus. If we contemplate a similar procedure on finite configurations, each act of the coalesce operator in a word $(\oplus\ominus)$ decreases the length of the configuration by one. These events are independent of all other occurrences.

We research the TACP below the action of the flip and coalesce operators(in this order), beginning from the initial measure $\delta_\ominus$. We aim to evaluate the asymptotic behavior of the TACP as time goes to infinity and to comprehend how the parameters α and β affect the evolution of the system. We denote the TACP by

$$\mu_t = \delta_\ominus(\mathsf{F}_\beta\mathsf{C}_\alpha)^t. \tag{1}$$

The following trivial cases in (1) are describe: if $\alpha = 0$ and $\beta \in (0, 1)$, then $\delta_{\ominus}(\mathsf{F}_\beta\mathsf{C}_\alpha)^t \to \delta_{\oplus}$ with $t \to \infty$; if $\beta = 1$ and $\alpha \in [0, 1]$, then $\delta_{\ominus}\mathsf{F}_\beta\mathsf{C}_\alpha = \delta_{\oplus}$ and if $\beta = 0$ and $\alpha \in [0, 1]$, then $\delta_{\ominus}\mathsf{F}_\beta\mathsf{C}_\alpha = \delta_{\ominus}$. However, we contemplate the case where $\beta \in (0, 1)$ and $\alpha \in (0, 1)$ only. This can only mean that there is a non-zero probability for the negative stimulus to turn into a positive stimulus and for two neighboring particles to merge into one with a negative stimulus.

Now, we shall state our main results.

Theorem 1. *For $\alpha \in (0, 1)$. If $\beta < \dfrac{\alpha^2}{55}$, then*

(A. 1) for all $t \in \mathbb{N}$, $\mu_t(\oplus) < 1$.

(B. 1) There is $\nu \in \mathcal{M}$ such that $\nu(\mathsf{F}_\beta\mathsf{C}_\alpha) = \nu$. Moreover $\nu(\oplus) < 1$.

As $\delta_{\oplus}$ is invariant for $\mathsf{F}_\beta\mathsf{C}_\alpha$, the Theorem 1 give us that μ_t is non-ergodic when $\beta < \alpha^2/55$.

Theorem 2. *For*

$$h(\beta) = \begin{cases} 2\beta, & se\ \beta \le \beta^*, \\ \dfrac{\beta}{(1-\beta)^2 + (1-\beta)\beta^2}, & se\ \beta > \beta^*, \end{cases}$$

where $\beta^* = -\dfrac{1}{6}\left(26 + 6\sqrt{33}\right)^{1/3} + \dfrac{4}{3\left(26 + 6\sqrt{33}\right)^{1/3}} + \dfrac{2}{3}$. *If $\alpha < h(\beta)$, then* $\lim_{t\to\infty} \mu_t = \delta_{\oplus}$.

In the above, Theorems 1 and 2 provide outcomes on the attitude of the measure μ_t as t approaches infinity, forming a kind of phase transition to μ_t. The first theorem states that if $\beta < \alpha^2/55$, then for all $t \in \mathbb{N}$, the frequency of positive stimuli in μ_t is less than 1 and there exists a measure ν that is invariant below the action of $\mathsf{F}_\beta\mathsf{C}_\alpha$ and has a frequency of positive stimuli that is lower than 1. The second theorem states that if $\alpha < h(\beta)$, then the measure μ_t converges to $\delta_{\oplus}$, meaning that the frequency of positive stimuli approaches 1 as t approaches infinity. In Figure 1 we summarize this outcome and some others from the numerical research.

Theorem 3. *Given $\alpha, \beta \in (0, 1)$ and $\mu \in \mathcal{M}_{\{\ominus,\oplus\}}$. If $\mu(\ominus) \in \left(0, \dfrac{\beta}{\alpha(1-\beta)}\right)$, then* $\lim_{t\to\infty} \mu\left(\mathsf{F}_\beta\mathsf{C}_\alpha\right)^t = \delta_{\oplus}$.

Theorem 4. *Let $s(\beta, \alpha)$ the supremum of density of $\oplus$ in μ_t. For every $\alpha \in \left(\dfrac{\beta}{(1-\beta)}, 1\right)$, $s(\beta, \alpha)$ is not continuous as a function of β.*

Theorems 3 and 4 show the behavior of the TACP model for different values of α and β. Theorem 3 states that, if the initial measure μ satisfies a certain condition, then the procedure $\mu\left(\mathsf{F}_\beta\mathsf{C}_\alpha\right)^t$ converges to the state where all particles are in the $\oplus$ state. Theorem 4 states that the function that measures the density of $\oplus$ in μ_t is not continuous as a function of β for a certain range of α. This implies that the TACP exhibits a kind of first-order phase transition.

3 Numerical study

The mean-field theory and Monte Carlo simulation are widely employed methods in scientific study to help guide the formulation of hypotheses that can then be formally proved. In the case of the TACP, the mean-field theory is employed to explain the dependence of the density of $\oplus$ in the measure $\mu\,(\mathsf{F}_\beta\mathsf{C}_\alpha)$ on the density of $\oplus$ in the measure μ. The Monte Carlo simulation is employed toguess the density of $\oplus$ in $\mu\,(\mathsf{F}_\beta\mathsf{C}_\alpha)$. By combining these two methods, studies can gain insight into the behavior of the TACP and make predictions that can then be rigorously proved by employing formal mathematical methods[19].

The *Mean-Field operator*, denoted by $\mathsf{m} : \mathcal{M} \to \mathcal{M}$, maps the set of translation-invariant probability measures in $\mathcal{A}^{\mathbb{Z}}$ to itself. The density of $\oplus$ in the measure $\mu(\mathsf{m}\mathsf{F}_\beta\mathsf{C}_\alpha)$ based solely on the density of $\oplus$ in the measure μ, and this dependence can be given as:

$$f(x) = \frac{b - \alpha b(1-b)}{1 - \alpha b(1-b)}.$$

where x signifies the density of plus in the measure μ, $f(x)$ denotes the density of plus in the measure $\mu(\mathsf{m}\mathsf{F}_\beta\mathsf{C}_\alpha)$, and $b = \mu\mathsf{F}_\beta(\oplus) = (1-\beta)\mu(\oplus) + \beta$. Thus, the study of the operator $\mathsf{F}_\beta\mathsf{C}_\alpha$ can be lowered to the research of the operator $\mathsf{m}\mathsf{F}_\beta\mathsf{C}_\alpha$, which is equivalent to the dynamic system study $f : [0,1] \to [0,1]$ with parameters $\alpha, \beta \in [0,1]$.

In this scope, we say that f is ergodic if there is a unique fixed point x_0, and $\lim_{t\to\infty} f^t(x)$ converges to x_0 from any $x \in [0,1]$. If this is not so, then we say that f is non-ergodic. Hence, $\mathsf{m}\mathsf{F}_\beta\mathsf{C}_\alpha$ is ergodic if $\alpha < 4\beta$ and non-ergodic if $\alpha \geq 4\beta$. These behaviors are in qualitative agreement with the phase transition defined in Theorems 1 and 2.

Now, we consider the analog of the TACP in finite configurations. In this case, our method is a Markov Chain with a countable set $\Omega = \{\oplus, \ominus\}^{\mathbb{Z}_n}$ of states known *periodics*, where $\mathbb{Z}_n$ is the set of integers modulo n. For a periodic $x \in \Omega$, we represent by $|x|$ the number of components in the periodic, and their indices are remainders modulo $|x|$.

In our simulations, we begin with an initial periodic x consisting of 1000 particles in the negative stimulus. The integer time t elevates from 0 to a maximum of $100,000$. The periodic at time t is signified by x^t and its components are represented by x_i^t, where $i = 0, \ldots, |x^t| - 1$. We denote the density of $\oplus$ in x as $\mathsf{dens}(\oplus|x)$. We stop the simulation when $t = 100,000$ or there are no particles in negative stimulus in the periodic x^t.

We assign the estimated value to the Boolean variable E (representing ergodicity), where $E = yes$ if the last periodic x^t consists of no particles in negative stimulus, and $E = no$ otherwise. If $E = yes$, we interpret this as proposing that the procedure with the provided values of α and β is ergodic; if $E = no$, we interpret this as proposing that the procedure is non-ergodic.

In our simulations, we began with $\beta = 0.015$ and then update β by applying 0.001, repeating this until β reaches 1 or $E = yes$, proposing ergodicity. We conduct this cycle 5 times and record the arithmetic mean of the 5 values of β obtained. We do this for 986 values of α, particularly $\alpha_i = 0.001 \cdot i$ for $i = 15, \ldots, 1000$. The corresponding registered value of β is denoted by β_i, resulting in 986 pairs (α_i, β_i). The curve labeled M.C. in Figure 1 contains these plotted pairs. We contemplate these values of α and β beginning at 0.015 to prevent large numerical fluctuations for small β values.

We can see that the M.C. curve is not exactly a curve but rather a scattered disposition of points. Nonetheless, it gives an idea of the behavior of the procedure. In the parameter space, there is an ergodic area for $\alpha < h(\beta)$, where $h(\beta)$ is provided in Theorem 2. On the

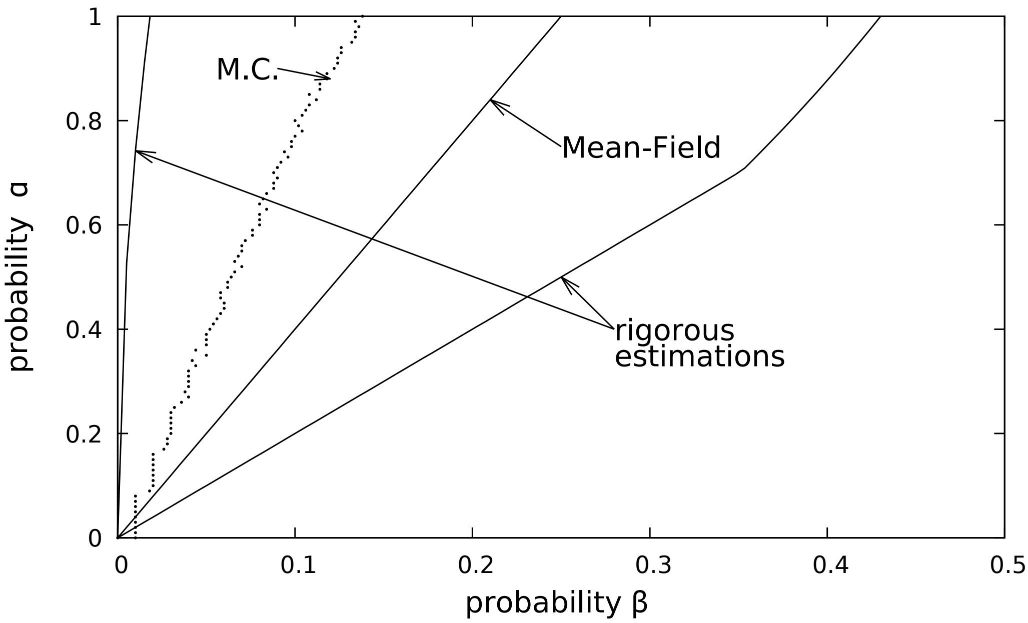

Figure 1: Here, we illustrate rigorous approximation and two estimations for the curve that split the ergodic and non-ergodic areas. Each point on the M.C. curve was gotten as an average of five independent experiments.

other hand, due to Theorem 1, there is a non-ergodicarea for $\alpha > \sqrt{55 \cdot \beta}$. Both areas are rigorously approximated in Figure 1.

As stated in Theorem 4, $s(\beta, \alpha)$ is the supremum of the density of pluses in μ_t. In this theorem, it is verified that $s(\beta, \alpha)$ is not continuous as a function of β. To gain a better comprehension of this quantity, we perform its numerical estimation. We define the estimator,

$$\overline{s(\beta, \alpha)} = \mathsf{dens}(\oplus | x^t)$$

for the final periodic x^t gotten in our simulation. The values of $\overline{s(\beta, \alpha)}$ gotten from our simulations are plotted in Figure 2. To offer a visual illustration of this data, each value is linked with a color, which is illustrated according to the rule revealed in the color box on the right.

The ergodic area, where $\overline{s(\beta, \alpha)} = 1$, is linked with the white color. In the non-ergodic area, all $\overline{s(\beta, \alpha)}$ values were lower than or equal to 0.25, which categorized its non-continuity as a function of β.

4 The operators of TACP

Here, we present the flip and coalesce operators.

The flip operator, F_β, is recognized. It flips the state of each component from $\ominus$ to $\oplus$ with probability β independently of the other components.

We describe the coalesce operator, C_α, as a combination of two operators: $\mathsf{C}_\alpha = \mathsf{Attraction}_\alpha \mathsf{Merge}$. First, $\mathsf{Attraction}_\alpha$ is used and then Merge. If a particle in a negative state is a right neighbor of a particle in a positive state, the two particles will experience a strong attraction toward each other with probability α or a weak attraction with probability $1 - \alpha$. This happens independently for each pair. If the attraction is strong, the left particle (in a positive state) will become neutral, and the Merge operator will merge the

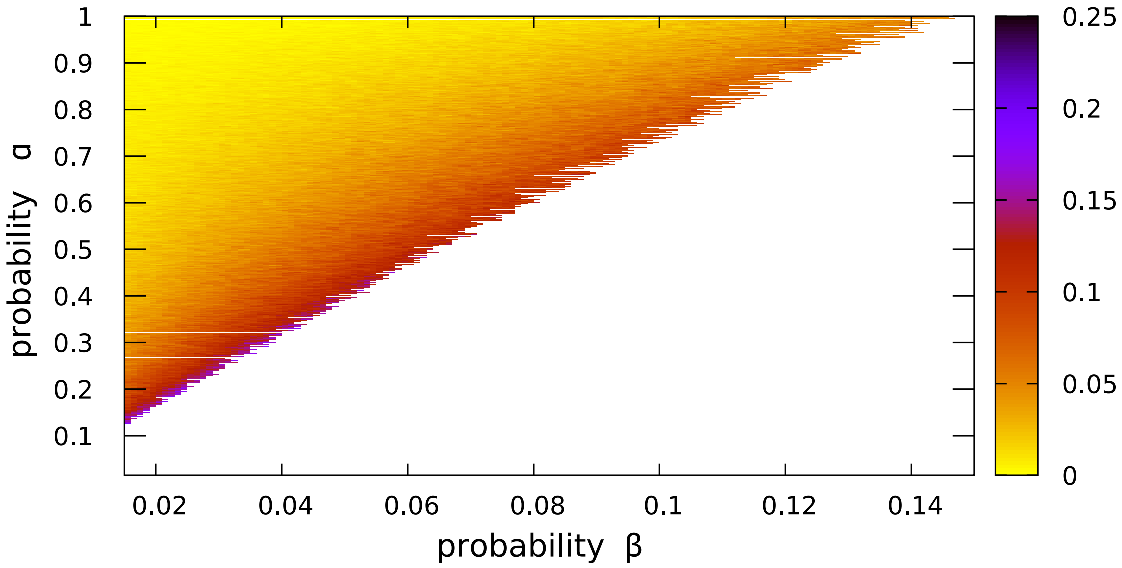

Figure 2: Here, colors were employed to signify the values of $\overline{s(\beta, \alpha)}$ in the region where the procedure is believed to be non-ergodic. The color box on the right demonstrates how colors from yellow to black denote the values of $\overline{s(\beta, \alpha)}$. When $\overline{s(\beta, \alpha)} = 1$, the associated color is white.

attracted particles in the neutral and negative states into a new particle with a negative state. The diagram below depicts the strong attraction (indicated by (i)) followed by the merge action (suggested by (ii)):

$$\oplus\ominus \xrightarrow{(i)} \odot\ominus \xrightarrow{(ii)} \ominus.$$

Let us define operator $\mathsf{Attraction}_\alpha$. This operator transform any measure on $\{\ominus, \oplus\}^{\mathbb{Z}}$ into a measure on $\{\ominus, \oplus, \odot\}^{\mathbb{Z}}$, where $\odot$ denotes a *neutral* stimulus.

From definition of $\mathsf{Attraction}_\alpha$, we have

$$\mu \mathsf{Attraction}_\alpha(\odot) = \alpha \mu(\oplus, \ominus) \leq \frac{1}{2} \tag{2}$$

Now, let us describe a variable-length operator $\mathsf{Merge} : \mathcal{M}_{\{\ominus,\oplus,\odot\}} \to \mathcal{M}_{\{\ominus,\oplus\}}$. For any non-empty word $W = (a_0, \ldots, a_k) \in dic(\ominus, \oplus)$, we define

$$\begin{aligned} \mu \mathsf{Merge}(W) &= \mu \mathsf{Merge}(a_0, \ldots, a_k) \\ &= \frac{1}{1 - \mu(\odot)} \sum_{n_1,\ldots,n_k=0}^{\infty} \mu(a_0 \odot^{n_1} a_1 \odot^{n_2} a_2 \cdots \odot^{n_{k-1}} a_{k-1} \odot^{n_k} a_k) \end{aligned} \tag{3}$$

where $\odot^n$ represents the word with n letters, every one of which is $\odot$, in particular $\odot^0 = \Lambda$. Notice that the formula (3) is non- linear, whence the well-developed theory of linear operators cannot be used here. Therefore, we get an extra challenge to deal with variable-length processes.

In Figure 3 we elucidate a possibility, i.e., that can occur with positive probability, the evolution of the one-time step of our approach.

5 Proof of Theorems 2, 3 and 4

From our major results, Theorem 1 necessitates many technical details. For this c, we decided to present its evidence in Section 6. Theorems 2, 3, and 4 only make sense when $\beta/\alpha(1-\beta) \leq 1$. We will assume this throughout the remainder of the proof.

The following identities will be beneficial in the proof:

$$\mu\mathsf{Merge}(\ominus) = \frac{\mu(\ominus)}{1-\mu(\odot)}, \qquad \mu\mathsf{Merge}(\oplus) = \frac{\mu(\oplus)}{1-\mu(\odot)}. \tag{4}$$

Proof of Theorem 3. Initially, we will show that $\mu\mathsf{F}_\beta\mathsf{C}_\alpha(\ominus) < \mu(\ominus)$. For all $\mu \in \mathcal{M}\{\ominus,\oplus\}$, it holds that $\mu\mathsf{Attraction}\alpha(\ominus) = \mu(\ominus)$. This, together with Equations (2) and (4), suggests that

$$\mu\mathsf{F}_\beta\mathsf{C}_\alpha(\ominus) = \frac{\mu\mathsf{F}_\beta(\ominus)}{1-\alpha\mu\mathsf{F}_\beta(\oplus,\ominus)}. \tag{5}$$

From the consistency of the measure, $\mu(\ominus) = \mu(\ominus,\ominus) + \mu(\ominus,\oplus)$. As $\mu(\ominus,\ominus) \geq 0$, it follows that $\mu(\ominus) \geq \mu(\oplus,\ominus)$.Therefore,

$$\frac{\mu\mathsf{F}_\beta(\ominus)}{1-\alpha\mu\mathsf{F}_\beta(\oplus,\ominus)} \leq \frac{\mu\mathsf{F}_\beta(\ominus)}{1-\alpha\mu\mathsf{F}_\beta(\ominus)}.$$

From the definition of F_β, it follows that $\mu\mathsf{F}_\beta(\ominus) = (1-\beta)\mu(\ominus)$. Hence,

$$\frac{\mu\mathsf{F}_\beta(\ominus)}{1-\alpha\mu\mathsf{F}_\beta(\oplus,\ominus)} \leq \frac{(1-\beta)\mu(\ominus)}{1-\alpha(1-\beta)\mu(\ominus)}. \tag{6}$$

Consequently, it suffices to illustrate

$$\frac{(1-\beta)\mu(\ominus)}{1-\alpha(1-\beta)\mu(\ominus)} < \mu(\ominus). \tag{7}$$

Indeed,

$$\begin{aligned}
&\frac{(1-\beta)\mu(\ominus)}{1-\alpha(1-\beta)\mu(\ominus)} < \mu(\ominus)\\
&\iff (1-\beta)\mu(\ominus) < \mu(\ominus) - \alpha(1-\beta)[\mu(\ominus)]^2\\
&\iff [-\beta + \alpha(1-\beta)\mu(\ominus)]\mu(\ominus) < 0\\
&\iff -\beta + \alpha(1-\beta)\mu(\ominus) < 0\\
&\iff \mu(\ominus) < \frac{\beta}{\alpha(1-\beta)}.
\end{aligned}$$

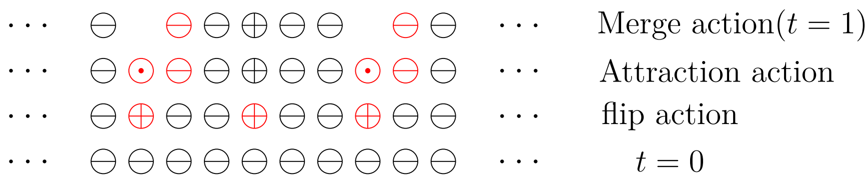

Figura 3: A fragment of the process and its evolution over one time step are shown. The red color suggests the location where the operator's action took place.

So, if $\mu(\ominus) \in \left(0, \frac{\beta}{\alpha(1-\beta)}\right)$, then, by observing (5), (6), and (7), we obtain

$$\mu \mathsf{F}_\beta \mathsf{C}_\alpha(\ominus) < \mu(\ominus). \tag{8}$$

Note that $\mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\ominus) > 0$, for each $t \in \mathbb{N}$. Now, we will show, using induction in t, that if $\mu(\ominus) \in \left(0, \dfrac{\beta}{\alpha(1-\beta)}\right)$, then $\mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\ominus) \in \left(0, \dfrac{\beta}{\alpha(1-\beta)}\right)$.

- The base case of the induction, $t = 1$, has been shown.
- According to our induction hypothesis, we have $\mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\ominus) \in \left(0, \dfrac{\beta}{\alpha(1-\beta)}\right)$. Thus, from (8),
$$\mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^{t+1}(\ominus) = \mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\mathsf{F}_\beta \mathsf{C}_\alpha)(\ominus) < \mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\ominus).$$

Therefore, the sequence $\mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\ominus)$ is monotonically decreasing and bounded, so shows that $\mu(\mathsf{F}_\beta \mathsf{C}_\alpha)^t(\ominus) \to 0$ as $t \to \infty$. □

Proof of Theorem 2.

If there is $t \in \mathbb{N}$ such that $\mu_t(\ominus) \in \left(0, \dfrac{\beta}{\alpha(1-\beta)}\right)$, then Theorem 2 follows from Theorem 3.

We will consider two cases.

Case 1. $\beta > \beta^*$, so $\alpha < \dfrac{\beta}{(1-\beta)^2 + (1-\beta)\beta^2}$. In this case, $\mu_1(\ominus) \in \left(0, \dfrac{\beta}{\alpha(1-\beta)}\right)$. Note the following chain of equalities:

$$\begin{aligned}
\mu_1(\ominus) &= \delta_\ominus \mathsf{F}_\beta \mathsf{C}_\alpha(\ominus) \\
&= \frac{\delta_\ominus \mathsf{F}_\beta(\ominus)}{1 - \alpha \delta_\ominus \mathsf{F}_\beta(\oplus, \ominus)} \\
&= \frac{1-\beta}{1 - \alpha \delta_\ominus \mathsf{F}_\beta(\oplus) \delta_\ominus \mathsf{F}_\beta(\ominus)} \\
&= \frac{1-\beta}{1 - \alpha\beta(1-\beta)}.
\end{aligned}$$

Additionally

$$\frac{1-\beta}{1-\alpha\beta(1-\beta)} < \frac{\beta}{\alpha(1-\beta)} \Longleftrightarrow \alpha < \frac{\beta}{(1-\beta)^2 + (1-\beta)\beta^2}.$$

Case 2. $\beta \leq \beta^*$, so $\alpha < 2\beta$.

We will demonstrate that it is impossible for $\mu_t(\ominus) \geq \dfrac{\beta}{\alpha(1-\beta)}$, for all $t \in \mathbb{N}$. Note that $\mu_{t+1}(\ominus) = \mu_t(\mathsf{F}_\beta \mathsf{C}_\alpha)(\ominus)$. From the definitions of F_β and C_α,

$$\mu_{t+1}(\ominus) = \frac{(1-\beta)\mu_t(\ominus)}{1 - \alpha \mu_t \mathsf{F}_\beta(\oplus, \ominus)}. \tag{9}$$

Hence, since for all $\mu \in \mathcal{M}\{\ominus, \oplus\}$, $\mu(\oplus, \ominus) \leq 1/2$, using (9),

$$\mu t + 1(\ominus) \leq \frac{(1-\beta)\mu_t(\ominus)}{1 - \alpha/2}.$$

Consequently,

$$\begin{aligned}\mu_{t+1}(\ominus) - \mu_t(\ominus) &\leq \frac{(1-\beta)\mu_t(\ominus)}{1-\alpha/2} - \mu_t(\ominus) \\ &= \frac{(1-\beta)\mu_t(\ominus) - \mu_t(\ominus) + (\alpha/2)\mu_t(\ominus)}{1-\alpha/2} \\ &= \frac{(\alpha/2-\beta)\mu_t(\ominus)}{1-\alpha/2}.\end{aligned}$$

This last equality signifies a linear function in regards to $\mu_t(\ominus)$. In addition, this linear function takes on negative values at both extreme points within its interval of definition. Therefore,

$$\mu_{t+1}(\ominus) - \mu_t(\ominus) \leq m,$$

Where m is a negative constant. Hence,

$$\mu_t(\ominus) - \mu_0(\ominus) \leq tm.$$

Therefore, $\mu_t(\ominus)$ approaches $-\infty$ as t approaches infinity, which is impossible.

□

Assuming Theorem 1 has been verified, we will now prove Theorem 4.

Proof of Theorem 4. It is not possible for $s(\beta, \alpha)$ to take values in the range $\left(1 - \frac{\beta}{\alpha(1-\beta)}, 1\right)$. If $s(\beta, \alpha)$ takes such a value, then there exists a natural number t such that $\mu_t(\oplus) > 1 - \frac{\beta}{\alpha(1-\beta)}$, which is equivalent to $\mu_t(\ominus) < \frac{\beta}{\alpha(1-\beta)}$. Accordingly, by Theorem 3, $\mu_t(\oplus)$ to 1 as $t \to \infty$. Hence, $s(\beta, \alpha) = 1$.

Based on Theorem 2, we must have $h(\beta) < 1$, but this condition is satisfied when $\beta < \bar{\beta}$, where $\bar{\beta} = 1/3\left(2 - 5\sqrt[3]{\frac{2}{3\sqrt{69}-11}} + \sqrt[3]{\frac{3\sqrt{69}-11}{2}}\right)$.

We will use this condition for β throughout this proof. Consequently, for any $\alpha \in \left(\frac{\beta}{1-\beta}, 1\right)$, $s(\beta, \alpha)$: (i) Tends to 0 as $\beta \to 0$ according to Theorem 1; (ii) is equal to 1 if $\alpha < h(\beta)$ as per Theorem 2; (iii) Cannot take values in the range $\left(1 - \frac{\beta}{\alpha(1-\beta)}, 1\right)$ because of Theorem 3. Thus, $s(\beta, \alpha)$ cannot be continuous as a function of β. □

6 A fixed-length representation to the TACP

We will present a fixed-length procedure where particles with neutral stimulus will not be merged at every time step. Rather, they will remain unaltered. Nevertheless, this causes the loss of locality. The space coordinate is represented by $x \in \mathbb{Z}$, and we use $y \in \mathbb{N}$, which is initially equal to zero and elevates by one after each action of F_β or C_α. Therefore, $y = 2t$ in the formula (1).

Now, we define the operator $\mathsf{P}_{\text{fixed}}$, represented by $\mathsf{P}_{\text{fixed}} = \overline{\mathsf{F}}_\beta \circ \overline{\mathsf{C}}_\alpha$. Here, $\overline{\mathsf{F}}_\beta$ and $\overline{\mathsf{C}}_\alpha$ are maps from $\mathcal{M}_{\{\ominus,\oplus,\odot\}}$ to $\mathcal{M}_{\{\ominus,\oplus,\odot\}}$. Informally, based on a configuration from $\{\ominus, \oplus, \odot\}^{\mathbb{Z}}$, the operator $\overline{\mathsf{F}}_\beta$ acts similarly to F_β. The operator $\overline{\mathsf{C}}_\alpha$ functions as follows: every word of the form $\oplus \odot^n \ominus$ becomes $\odot^{n+1}\ominus$ with probability α, and these transformations are independent of each other. We consider $\odot^0 = \Lambda$, which signifies the empty word.

We signify by

$$V = \{(x, y), x \in \mathbb{Z}, y \in \mathbb{Z}_+\}.$$

Let $V_y \subset V$ such that the second coordinate of each pair in V is equal to y. We refer to V_y as the $y-$levels or simply the levels. Each pair $(x, y) \in V$ has an associated random variable $\eta_y(x) \in \{\ominus, \oplus, \odot\}$. The distribution of $\eta_y(x)$ is represented by $\delta_\ominus \mathsf{P}^y_{\mathsf{fixed}}$. So, if $\eta_y(x) = a \in \{\ominus, \oplus, \odot\}$, it has distribution $\delta_\ominus \mathsf{P}^y_{\mathsf{fixed}}(x = a)$. It is also expressed that $\eta_0(x) = \ominus$ for all $x \in \mathbb{Z}$.

Under $\mathsf{P}_{\mathsf{fixed}}$, particles in a neutral stimulus to the left of a particle in a negative stimulus will never merge and all particles will preserve their original integer indices. For each positive integer t, when y is even, i.e. $y = 2t$, the operator $\overline{\mathsf{F}}_\beta$ is used and when y is odd, i.e. $y = 2t + 1$, the operator $\overline{\mathsf{C}}_\alpha$ is applied.

We represent by ν the measure on $\{\ominus, \oplus, \odot\}^V$ and

$$\nu(\mathsf{P}_{\mathsf{fixed}})^y = \nu_y \text{ for all } y \in \mathbb{Z}_+.$$

From the aforementioned, we can associate the processes ν_t and μ_t as follows:

$$\nu_{2t}\mathsf{Merge} = \mu_t, \qquad \text{para todo } t \in \mathbb{Z}_+. \tag{10}$$

In fact, (10) can be verified by induction.

Items (a) - (d) are easy to check.

$$\begin{cases} \text{(a)} & \nu\left(\eta_y(x) = \ominus\right) > 0 \text{ for all } (x, y) \in V. \\ \text{(b)} & \text{For any } x_0 \in \mathbb{Z} \text{ and } y \in \mathbb{N} \\ & \nu\left(\forall x \geq x_0 : \eta_y(x) \neq \ominus\right) = \nu\left(\forall x \leq x_0 : \eta_y(x) \neq \ominus\right) = 0. \\ \text{(c)} & \mu_t(\ominus) > 0 \quad \forall \quad t \in \mathbb{N}. \\ \text{(d)} & \text{For any } x_0 \in \mathbb{Z} \text{ and } y \in \mathbb{N} \\ & \mu_t(\forall x \geq x_0 : s_x \neq \ominus) = \mu_t(\forall x \leq x_0 : s_x \neq \ominus) = 0. \end{cases} \tag{11}$$

In the following text, we will only consider events with non-zero probability. We will describe a planar graph G from the set $\{\ominus, \oplus, \odot\}^V$. Vertices in G correspond to pairs $(x, y) \in V$ such that $\eta_y(x) \neq \odot$. The set of these vertices is signified by V_G. Each vertex (x, y) is positioned such that the x axis is horizontal and the y axis is vertical.

The graph G has two types of edges: vertical and horizontal. For vertices (x, y_1) and (x, y_2) with $y_2 - y_1 = 1$, a vertical edge connects them. The direction from (x, y_1) to (x, y_2) is referred to as *north,* and the opposite direction is called *south.* (x, y_1) is the south neighbor of (x, y_2) and (x, y_2) is the north neighbor of (x, y_1).

The graph G has horizontal edges connecting vertices (x_1, y) and (x_2, y) if $x_1 < x_2$ and all x between x_1 and x_2 satisfy $\eta_y(x) = \odot$. The direction of the edge from (x_1, y) to (x_2, y) is referred to as *east* and the other direction is *west.* (x_1, y) is the west neighbor of (x_2, y), and (x_2, y) is the east neighbor of (x_1, y). The edges are denoted by straight lines connecting the points representing the end of the edge. With this, we have described the graph G.

6.1 A planar representation to G and its dual $\overline{G}$

The graph G is represented in the plane as a *picture.* Each vertex of G can be either a $\oplus$-vertex or a $\ominus$-vertex. The picture is divided into regions called *faces,* which are all closed and have a common edge with their neighbors. Bounded faces are referred to as *boxes.* The only unbounded face is the bottom half of the plane.

A box is a rectangle between two parallel lines at levels y_1 and y_1+1, formally expressed as:

$$\{(x, y) \in \mathbb{R}^2 : x_1 \leq x \leq x_2, y_1 \leq y \leq y_1 + 1\}. \tag{12}$$

For even y_1, the box (12) has one south neighbor as there are no vertices on its south wall. For odd y_1, there are two cases: if there was a coalesce between sites x_1 and x_2 from level y_1 to y_1+1, the box has two south neighbors. If there was no coalesce, the box has only one south neighbor. These elements can be viewed in Figure 4.

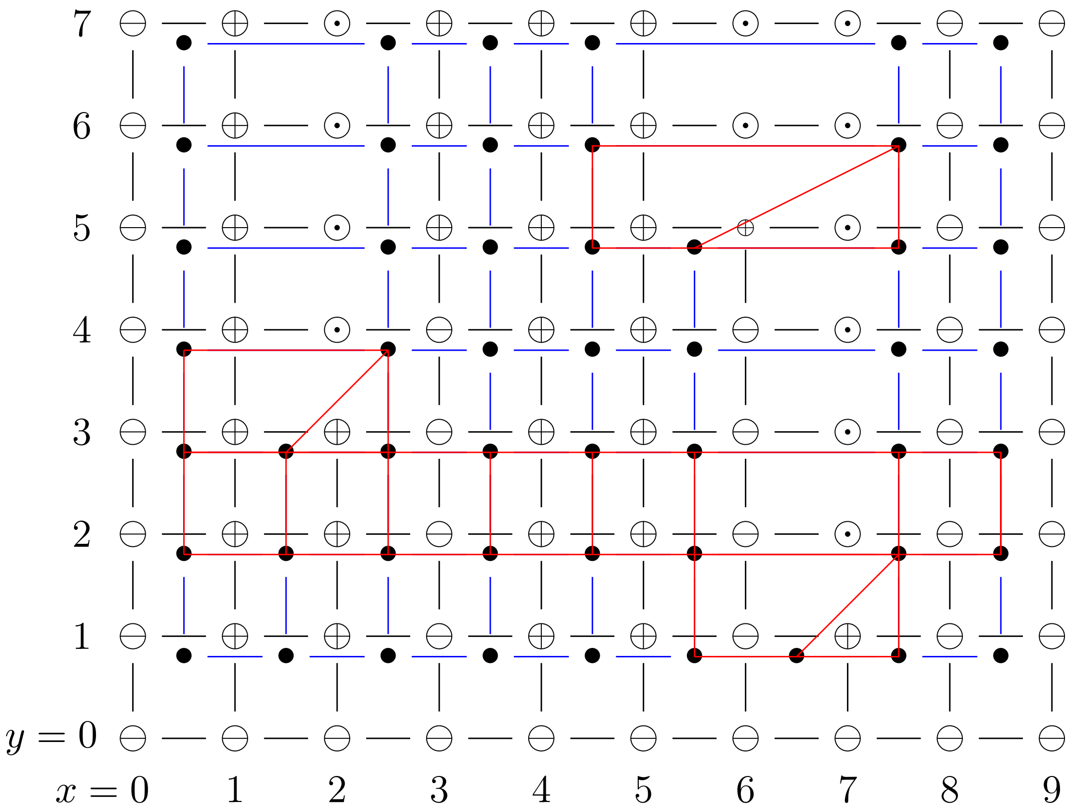

Figure 4: A fragment of the picture of G and its dual $\overline{G}$ is depicted. Some figures created in the dual are emphasized in red and are called rectangles, trapezoids, and triangles.

Let $\overline{G}$ be the dual of G. We will now elucidate the picture of this dual. We place the vertex of $\overline{G}$, which is the dual of the box (12), at the point:

$$\left(x_2 - \frac{1}{2}, y_1 + 1 - \varepsilon\right), \tag{13}$$

where $\varepsilon > 0$ is chosen differently for each box. The corresponding ε can be selected small enough for each box.

We will refer to the vertex (13) as being in a sub-level $y_1 + 1$. If $y_1 + 1$ is even, we will say that it is at a sub-even level. However, if $y_1 + 1$ is odd, we will say that it is in a sub-odd level.

For the unbounded face in G, there is a single vertex in $\overline{G}$ that will be positioned appropriately far in the negative y direction. The edges leading to it are rays in the same direction. For other edges, we denote them with straight segments connecting the points that represent their ends.

With these definitions, we have defined the picture of $\overline{G}$. Unbounded faces of $\overline{G}$ correspond to vertices of G at level zero. A face of $\overline{G}$ is called a west (or east, north, or south) neighbor of another face of $\overline{G}$ if their corresponding vertices of G are related similarly.

Based on what was stated about the vertices of G at odd levels, any face of the dual graph $\overline{G}$ at an odd level has at most one north neighbor. If it has one, it is referred to as a *rectangle* or a *trapezium*; otherwise, it is called a *triangle*. These faces are approximately rectangles, trapeziums, and triangles. In Figure 4, we highlight, in red, the faces of $\overline{G}$ that are approximately trapeziums and triangles in sub-levels 2, 4, and 6, and the faces of $\overline{G}$ that we name rectangles in sub-level 3. Nevertheless, we can detect that there are faces of $\overline{G}$ that are rectangles even in even sub-levels. The black lines signify a fragment of the picture of G, while the picture of its dual $\overline{G}$ is a combination of red and blue colors.

7 An upper bound to $\mu_T(\oplus)$

Let $T \in \mathbb{N}$ be arbitrary. By (11), $\mu_T(\ominus)$ is positive, therefore the fraction $\frac{\mu_T(\oplus)}{\mu_T(\ominus)}$ is well-defined. We aim to prove:

$$\mu_T(\oplus) = \sum_{k=1}^{\infty} \mu_T(\ominus, \oplus^k). \tag{14}$$

Let $\omega \in \{\ominus, \oplus\}^{\mathbb{Z}}$ be a configuration and consider the event of finding a $\oplus$ at a position. We can section them according to the number of particles with positive stimuli to the left of that position. By (11), this number is finite almost surely, hence

$$\mu_T(\oplus) \leq \frac{\mu_T(\oplus)}{\mu_T(\ominus)} = \sum_{k=1}^{\infty} \frac{\mu_T(\ominus, \oplus^k)}{\mu_T(\ominus)}. \tag{15}$$

We will now derive a new identity for the last term on the right side of (15). Let

$$\Omega_0 = \{\omega \in \{\ominus, \oplus, \odot\}^V : \eta_{2T}(0) = \ominus\}$$

and for $\omega \in \Omega_0$ define

$$x_{\max}(\omega) = \inf\{x > 0 : \eta_{2T}(x) = \ominus\}.$$

Employing (11), we conclude that $x_{\max}(\omega)$ exists almost surely.

For any $\omega \in \Omega_0$, we define the set

$$flowers = \{(x, 2T) : 0 < x < x_{\max}(\omega) \text{ and } \eta_{2T}(x) = \oplus\}.$$

The cardinality of $flowers$ is signified by $\phi(\omega)$. It has been demonstrated that $\phi(\omega)$ is finite almost surely.

For any natural number k, we define

$$\Omega_k = \{\omega \in \Omega_0 : \phi(\omega) \geq k\}.$$

It holds that $\Omega_{i+1} \subseteq \Omega_i$ for all $i \in \mathbb{N}$. Our goal is to prove (16),

$$\frac{\pi(\Omega_k)}{\pi(\Omega_0)} = \frac{\mu_T(\ominus, \oplus^k)}{\mu_T(\ominus)}. \tag{16}$$

For an arbitrary natural number T, $\pi(\Omega_0)$ is the probability of finding a particle in the minus state at position $(0, 2T)$. As ν is stimulated by π, computing $\pi(\Omega_0)$ is equivalent to calculating $\nu_{2T}(x_0 = \ominus)$. And as ν_{2T} is translation-invariant, we have $\nu_{2T}(x_0 = \ominus) = \nu_{2T}(\ominus)$, and thus $\pi(\Omega_0) = \nu_{2T}(\ominus)$. From (4) and (10),

$$\mu_T(\ominus) = \nu_{2T}\mathsf{Merge}(\ominus) = \frac{\nu_{2T}(\ominus)}{1 - \nu_{2T}(\odot)} \Longrightarrow \pi(\Omega_0) = \nu_{2T}(\ominus) = \mu_T(\ominus)(1 - \nu_{2T}(\odot)). \quad (17)$$

If $\omega \in \Omega_k$, then the configuration at the level $2T$ has one of the words

$$\ominus \odot^{n_1} \oplus \odot^{n_2} \cdots \oplus \odot^{n_k - 1} \oplus \odot^{n_k} \oplus, \quad (18)$$

Where $\odot^n$ means the word $W = \odot \ldots \odot$ where $|W| = n$. Therefore, compute $\pi(\Omega_k)$ means calculate the probability of find (18) in the level $2T$ by measure ν because, it is translation-invariant. So,

$$\pi(\Omega_k) = \sum_{n_1, \ldots, n_k = 0}^{\infty} \nu_{2T}(\ominus \odot^{n_1} \oplus \cdots \odot^{n_k} \oplus).$$

In addition, from (3) and (10),

$$\begin{aligned} \mu_T(\ominus, \oplus^k) &= \nu_{2T}\mathsf{Merge}(\ominus, \oplus^k) \\ &= \frac{1}{1 - \nu_{2T}(\odot)} \sum_{n_1, \ldots, n_k = 0}^{\infty} \nu_{2T}(\ominus \odot^{n_1} \oplus \cdots \odot^{n_k} \oplus). \end{aligned}$$

So,

$$\mu_T(\ominus, \oplus^k) = \frac{\pi(\Omega_k)}{1 - \nu_{2T}(\odot)}.$$

Thus,

$$\pi(\Omega_k) = \mu_T(\ominus, \oplus^k)(1 - \nu_{2T}(\odot)).$$

Dividing this by (17), we prove (16). So,

$$\sum_{k=1}^{\infty} \frac{\pi(\Omega_k)}{\pi(\Omega_0)} = \sum_{k=1}^{\infty} \frac{\mu_T(\ominus, \oplus^k)}{\mu_T(\ominus)},$$

and using (15),

$$\frac{\mu_T(\oplus)}{\mu_T(\ominus)} = \sum_{k=1}^{\infty} \frac{\pi(\Omega_k)}{\pi(\Omega_0)}.$$

7.1 The contours

Consider any $\omega \in \Omega_1$. Let us call a path in G where every step goes north or west a *north-west* path. A vertex in G is called a *root* if starting from this vertex there is a north-west path to a flower such that all the vertices in this path are $\oplus$-vertices.

Since T is fixed, $x_{\max}(\omega)$ exists almost surely, and the set of flowers is finite almost surely, and hence the set of roots is also finite almost surely.

We call the faces of $\overline{G}$ that are dual to roots *dual roots*, and the union of these dual roots is represented by U. Since each dual-root is bounded and we presume all faces to be closed, U is also bounded and closed, and since the set of roots is finite (which is easy to prove), it follows that U is homeomorphic to a closed disk. Therefore, the border of U is a closed curve that incorporates the north end of the side, V_0.

Thus, we consider that this curve starts and ends at V_0 and walks around U in a counter-clockwise direction. This curve can be represented as a path in $\overline{G}$, which we denote by $\mathsf{tour}(\omega)$. Figure 5 illustrates such a path.

Now, we will categorize all the steps that $\mathsf{tour}(\omega)$ may include in $\overline{G}$. Firstly, we will classify steps in G. We name the elements of the set

$$\{1, 1', 2, 2', 2'', 3, 4, 4', 5\}$$

types.

Let $(F_x^t)_{(x,t)\in\mathbb{Z}\times\mathbb{Z}+}$ and $(M_x^t)_{(x,t)\in\mathbb{Z}\times\mathbb{Z}+}$ be two arrays of independent and identically distributed random variables. F_x^t has a Bernoulli distribution with parameter β, and is a map $F_x^t : \{move, stay\} \to \{0, 1\}$, where $F_x^t(move) = 1$ and $F_x^t(stay) = 0$. M_x^t possesses a Bernoulli distribution with parameter α, and is a map $M_x^t : \{strong, weak\} \to \{0, 1\}$, where $M_x^t(strong) = 1$ and $M_x^t(weak) = 0$.

Step in G beginning at a $\oplus$-vertex	Type	Associated event	Associated variable
Step west at an even level	1	trivial	none
Step west at an odd level	1'	trivial	none
Step from $(x, 2t+1)$ to $(x, 2t)$ if $F_x^t = 1$	2	$F_x^t = 1$	F_x^t
Step from $(x, 2t+1)$ to $(x, 2t)$ if $F_x^t = 0$	2'	$F_x^t = 0$	F_x^t
Step south from an even to an odd level	2"	trivial	none
Step from $(x, 2t+1)$ to its east neighbor if $M_x^t = 1$	3	$M_x^t = 1$	M_x^t
Step from $(x, 2t+1)$ to its east neighbor if $M_x^t = 0$	4	$M_x^t = 0$	M_x^t
Step east at an even level	4'	trivial	none
Step north	5	trivial	none

Table 1: Definition of types, associated events and associated variables with steps in G starting at a $\oplus$-vertex.

In Table 1, we list trivial and non-trivial events. The trivial events are linked to $\{\ominus, \oplus, \odot\}^V$ and are called trivial. The non-trivial events are defined by their conditions. Every step in G that has a type also has a *chance*, which is equal to the chance introduced in Table 2. The same one-to-one correspondence between steps in G and steps in $\overline{G}$ will be employed. If an edge $\bar{e}$ in $\overline{G}$ is the dual of an edge e in G, then the dual direction of $\bar{e}$ is to the left of the direction of e when moving along e in the given direction. A step in $\overline{G}$ has a type if and only if the left side of the step is a $\oplus$-face.

Table 2 demonstrates the *shifts* defined for all types. The component of shifts is represented HS and VS (abbreviations for horizontal shift and vertical shift).

Each step in $\mathsf{tour}(\omega)$ has a type. It can be decomposed into two paths: the first one is referred to as the *bag* and represented $\mathsf{bag}(\omega)$. In this path, all the types of steps are different from 5. The second path is named the *lid* and depicted $\mathsf{lid}(\omega)$. All its steps have a type of 5 and it has exactly $\phi(\omega)$ steps. These easily acceptable statements can be validated by making a small adaptation of Lemma 3 in [2].

Step in $\overline{G}$ having a $\oplus$-face on its left side	Tipe	Chance	Shift
Step south across an even level	1	1	$(0,-1)$
Step south across an odd level	1’	1	$(0,-1)$
“$F_x^t = 1$” step east at a sub-level odd	2	β	$(1,0)$
“$F_x^t = 0$” step east at a sub-level even	2’	$1-\beta$	$(1,0)$
Step east at a sub-level even	2”	1	$(1,0)$
“$M_x^t = 1$” step north across an odd level	3	α	$(0,1)$
“$M_x^t = 0$” step north across an odd level	4	$1-\alpha$	$(0,1)$
Step north across an even level	4’	1	$(0,1)$
Step west	5	1	$(-1,0)$

Table 2: Definition of types, chances and shift of steps in $\overline{G}$ with a $\oplus$-face on its left side.

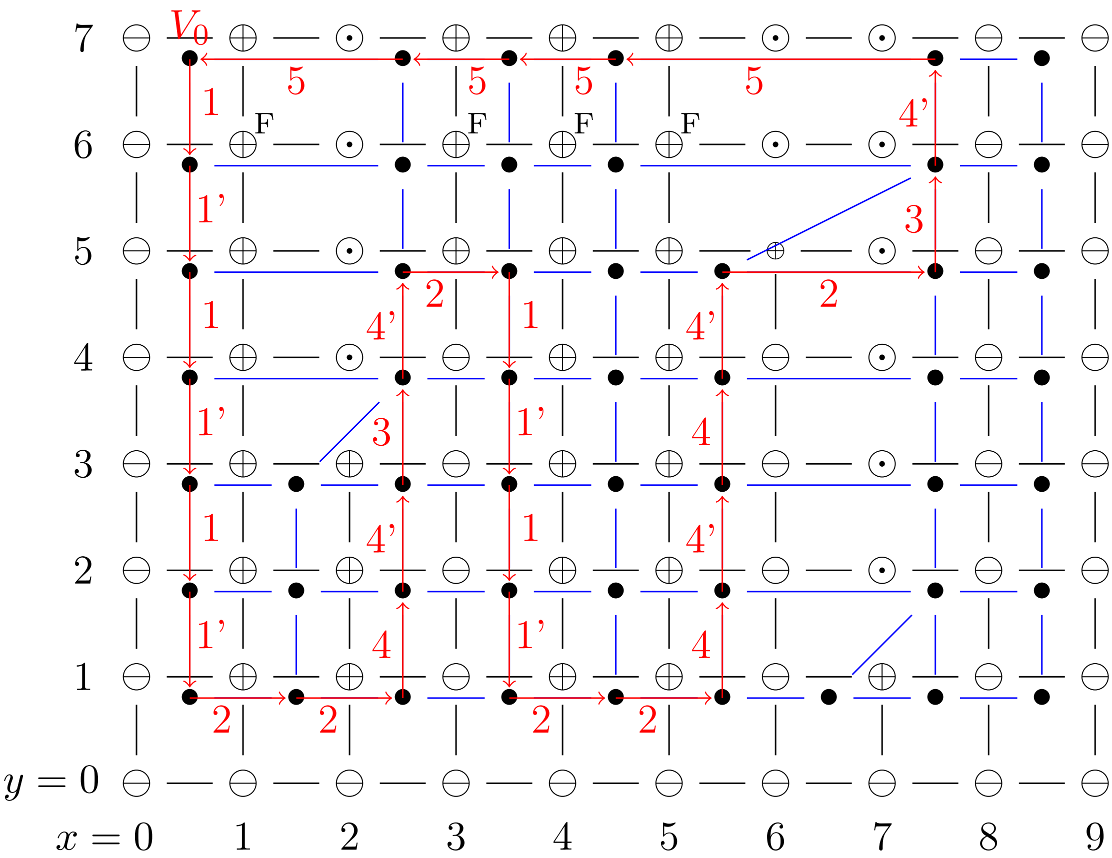

Figure 5: Illustration of a fragment of the graph G, its dual $\overline{G}$, and a tour represented by arrows and their respective types.

7.2 Paths and codes

We will express any sequence of types as a *code*. The sum of shifts of its terms is called the *shift of the code*, and the product of the chances of its terms is referred to as the *chance of the code*.

Given a path, p, where all its steps have types, the sequence of these types is known as the "code of p,"represented $\mathsf{code}(p)$. Consider a path in $\overline{G}$ starting at V_0, where all its steps have types and all the basic variables linked to its steps are independent of each other and from Ω_0. This path is called *well-placed.*

For some $\omega \in \Omega_0$ and a code C, we say that ω *realizes* C if $\overline{G}$ comprises a well-placed path p such that the code of p is equal to C.

$$\mathsf{real}(C) = \{\omega \in \Omega_0 : \omega \textit{ realizes } C\}.$$

Lemma 1. *For any code C we have,*

$$\frac{\pi(\mathsf{real}(C))}{\pi(\Omega_0)} \leq \mathsf{chance}(C). \tag{19}$$

Proof. Let C be a code of length n, i.e., $C = (c_1, \ldots, c_n)$, then $\mathsf{real}(C)$ is the set of $\omega \in \Omega_0$ which realizes C. If there is no $\omega \in \Omega_0$ which realizes C, then $\mathsf{real}(C) = \emptyset$ and therefore (19) is satisfied.

We assume $\mathsf{real}(C) \neq \emptyset$. If C_1 and C_2 are two codes such that $C_1 \neq C_2$, p_1 and p_2 are two well-placed paths such that $\mathsf{code}(p_1) = C_1$ and $\mathsf{code}(p_2) = C_2$, then $p_1 \neq p_2$. As follows, for all $\omega \in \mathsf{real}(C)$, there is a unique path p such that $\mathsf{code}(p) = C$. Thus, computing $\pi(\mathsf{real}(C))$ is equal to calculating the probability of occurring of the path p such that $\mathsf{code}(p) = C$. That is, $\pi(\mathsf{real}(C)) = \mathbb{P}(p)$. Where, $\mathbb{P}(p)$ is the probability of happening on the path p.

Let be $\omega \in \mathsf{real}(C)$, then the dual graph $\overline{G}$ comprises a well-placed path p, such that $\mathsf{code}(p) = C$. Since p is a well-placed path, it starts at V_0 and all of its steps have types. Let $p_1, \ldots, p_n$ the steps of p. To p_i we associate the type c_i, for each $i = 1, \ldots, n$, and we take the probability of occurrence of p_i, given by the basic variables linked to the steps of the path p, smaller than or equal to $\mathsf{chance}(c_i) \cdot \pi(\Omega_0)$. Since p is well-placed, the basic variables related to your steps are independent of each other and Ω_0, so

$$\begin{aligned} \mathbb{P}(p) &= \mathbb{P}(p_1) \cdot \ldots \cdot \mathbb{P}(p_n) \\ &\leq \mathsf{chance}(c_1) \cdot \ldots \cdot \mathsf{chance}(c_n) \cdot (\pi(\Omega_0))^n \\ &= \mathsf{chance}(C) \cdot (\pi(\Omega_0))^n. \end{aligned} \tag{20}$$

Hence, since $0 < \pi(\Omega_0) < 1$ and using (20) we have that

$$\frac{\pi(\mathsf{real}(C))}{\pi(\Omega_0)} \leq \frac{\mathbb{P}(p)}{(\pi(\Omega_0))^n} \leq \mathsf{chance}(C).$$

□

An outcome that can easily be adapted from Lemma 4 in [2] is: *Every $\omega \in \Omega_1$ realizes the code of* $\mathsf{bag}(\omega)$. Using this outcome along with Lemma 1, for each natural number k, we can acquire:

$$\frac{\pi(\Omega_k)}{\pi(\Omega_0)} \leq \frac{\sum \pi(\mathsf{real}(\mathsf{code}(\mathsf{bag}(\omega))))}{\pi(\Omega_0)} \leq \sum \mathsf{chance}(\mathsf{code}(\mathsf{bag}(\omega))), \tag{21}$$

Where both sums are taken over all diverse $\mathsf{code}(\mathsf{bag}(\omega))$ for $\omega \in \Omega_k$. To estimate the last sum, for each natural number $k \in \mathbb{N}$, we describe a set of codes, represented by LC_k, and refer to its elements as *k-legal codes*. A code $C = (c_1, \dots, c_n)$ belongs to LC_k if it satisfies the following conditions:

$$\begin{cases} (LC-a) & c_1 = 1 \text{ and } c_n = 4'. \\ (LC-b) & \text{All the terms of } C \text{ belong to the list } 1, 1', 2, 3, 4, 4'. \\ (LC-c) & \text{All the pairs } (c_i, c_{i+1}) \text{ belong to the list} \\ & 11', 1'1, 1'2, 21, 22, 23, 24, 34', 44', 4'2, 4'3, 4'4 \\ (LC-d) & \mathsf{HS}(C) \geq k \text{ and } \mathsf{VS}(C) = 0. \end{cases} \tag{22}$$

Note that for all $k \geq 1$, $LC_k \supseteq LC_{k+1}$. We represent LC_1 as LC and designate its elements simply as *legal codes*. The definition of legal codes is chosen to fit the code of $\mathsf{bag}(\omega)$ as shown in Lemma 2.

Lemma 2. *For all* $\omega \in \Omega_k$*, the code of* $\mathsf{bag}(\omega)$ *belongs to* LC_k*.*

Proof. Let us prove that the code of $\mathsf{bag}(\omega)$ satisfies all the conditions of (22).
Proof of condition $(LC-a)$*.* The code of $\mathsf{bag}(\omega)$ starts with type 1, because $\mathsf{bag}(\omega)$ begins from the east side of the dual face to the vertex $(0, 2T)$, going from north to south, that is, it is a south step passing through an even level, therefore, based on Table 2, its type is 1. The code of $\mathsf{bag}(\omega)$ ends with type 4', because as the flowers are at an even level, the last step of $\mathsf{bag}(\omega)$ is a step north through an even level, so it type is 4'.
Proof of condition $(LC-b)$*.* The code of $\mathsf{bag}(\omega)$ can not hold type 5. It remains for us to indicate that $\mathsf{code}(\mathsf{bag}(\omega))$ can not contain types 2' or 2". Suppose some step of $\mathsf{bag}(\omega)$ has type 2' or 2", then this step has a root on its left side, which is north. But if the steps are type 2' or 2", then on its south side it also has a root, because there is a northwest path from this vertex (that is in state $\oplus$) for some flower, which is impossible since the steps in $\mathsf{tour}(\omega)$ cannot separate the roots from one another.
Proof of condition $(LC-c)$*.* Let us introduce numerous arguments, owing to which all the combinations of types (c_i, c_{i+1}), not included in our list, are impossible in the code of $\mathsf{bag}(\omega)$.

- Pairs, in which the first term is in the set $\{1, 3, 4\}$ and the second term is in the set $\{1, 2, 3, 4\}$, are impossible because the first term ends at an even sub-level, but the second term starts at an odd sub-level.

- Pairs, in which the first term is in the set $\{1', 2, 4'\}$ and the second term is in the set $\{1', 4'\}$, are impossible, because the first term ends at an odd sub-level, but the second term starts at an even sub-level.

- Pairs, in which the first term is in the set $\{1, 1'\}$ and the second term is in the set $\{3, 4, 4'\}$, are impossible. If the pairs gotten from these sets could happen, then as the first type is a south pass and the second type is a north pass, they would be on the same vertical edge of $\overline{G}$. But the terms of the first set need a $\oplus$-face on its east side, while the terms of the second set need a $\ominus$-face on its east side because If $\mathsf{bag}(\omega)$ comprises of a step type 3, 4 or 4', then this step has a $\ominus$-face on its right (that is, east) side, which is absurd.

- Pairs, where the first term is in the set $\{3, 4\}$ and the second term is 1' are impossible because they have to be on the same edge, but the face on the east side of type 3 or 4 steps must be in the state $\ominus$, while the face on the east side of type 1' step must be in the state $\oplus$.

- Pairs, where the first term is 4' and the second term 1, are impossible because they have to be on the same edge, but the face on the east side of type 4' step must be in the state $\ominus$, while the face on the east side of type 1 step must be in the state $\oplus$.

Proof of condition $(LC-d)$. We have that the path $\mathsf{tour}(\omega)$ is a concatenation of the paths $\mathsf{bag}(\omega)$ and $\mathsf{lid}(\omega)$, where $\mathsf{lid}(\omega)$ has $\phi(\omega)$ steps type 5, all of which have shift $(-1,0)$, then the shift of $\mathsf{lid}(\omega)$ is $(-\phi(\omega),0)$. As follows, as the contour $\mathsf{tour}(\omega)$ begins and ends in V_0, follow that $\mathsf{bag}(\omega)$ has x east steps, where $x \geq \phi(\omega)$ (if there are not murders $x=\phi(\omega)$). From condition $(LC-b)$ the only east step that $\mathsf{bag}(\omega)$ can have is the type 2, whose shift is $(1,0)$, then $\mathsf{HS}(\mathsf{bag}(\omega)) = x \geq \phi(\omega) \geq k$.

Since the contour $\mathsf{tour}(\omega)$ starts and ends at V_0 and the path $\mathsf{lid}(\omega)$ only has west steps (type 5), then the number of steps south (types 1 and 1') is equal to the number of steps north (types 3, 4 and 4') in the path $\mathsf{bag}(\omega)$. Let's call this number of steps north, therefore the number of steps south, y. Since all south types have shift $(0,-1)$ and all north types have shift $(0,1)$ follows that $\mathsf{VS}(\mathsf{bag}(\omega)) = -y+y=0$.

□

Using Lemma 2 and (21), we get

$$\frac{\pi(\Omega_k)}{\pi(\Omega_0)} \leq \sum_{C\in LC_k} \mathsf{chance}(C). \tag{23}$$

We will simplify our task by lowering the types to four major types to make our numerical estimation easier to visualize. The elements of the set $\{1,2,3,4\}$ will be called the *main types.* All the quantities defined for types are also valid for major forms, together with shifts and chances (as illustrated in Table 2).

A *main code* is a finite sequence comprising only major types. For any code C, the main code gotten from C by removing all non-main types is represented $\mathsf{short}(C)$. To simplify our task, we will deal with $\mathsf{short}(\mathsf{code}(\mathsf{bag}(\omega)))$ rather than $\mathsf{code}(\mathsf{bag}(\omega))$. For every natural number k, we define a set of codes called *k-legal main codes,* denoted by LMC_k. A k-legal main code is a main code $C=(c_1,\dots,c_n)$ that satisfies the following conditions:

$$\begin{cases} (LMC-a) & c_1 = 1. \\ (LMC-b) & \text{For every } i=1,\dots,n-1 \text{ it is impossible that} \\ & (c_i=1, c_{i+1}=3) \text{ or } (c_i=3, c_{i+1}=1) \text{ or} \\ & (c_i=1, c_{i+1}=4) \text{ or } (c_i=4, c_{i+1}=1). \\ (LMC-c) & c_n \text{ is equal to 3 or 4.} \\ (LMC-d) & \mathsf{HS}(C) \geq k \\ (LMC-e) & \mathsf{VS}(C) = 0. \end{cases} \tag{24}$$

Note that $LMC_k \supseteq LMC_{k+1}$ for all $k \geq 1$. We refer to LMC_1 as LMC and call the elements of LMC simply *legal main codes.* It is reported that any legal main code has a length of at least three, derived from $(LMC-a)$, $(LMC-b)$, and $(LMC-c)$.

From this point onward, we represented by C the legal main codes and by C' the legal codes. Hence, for any legal main code C, we will represent by $\mathsf{long}(C)$ the *legal code* C' such that $C = \mathsf{short}(C')$.

We will establish a bijection between the major legal codes and the legal codes. Let us represent this bijection by

$$\mathsf{long} : LMC \to LC.$$

Given $C \in LMC$, $\mathsf{long}(C) = C'$ is obtained through the following procedure:

$$\begin{cases} \text{(a)} & \text{We start with } C. \\ \text{(b)} & \text{After every 1 we insert 1'.} \\ \text{(c)} & \text{After every 3 we insert 4'.} \\ \text{(d)} & \text{After every 4 we insert 4'.} \end{cases} \tag{25}$$

The inverse of the long application is the short application, which takes a legal code C' and eradicates the types 1' and 4'.

We will estimate the sum on the right side of (23). Remember that given $C' \in LC_k$ there is unique $C \in LMC_k$ such that $C' = \mathsf{long}(C)$. From (25), C and C' vary by types 1' and 4', which has chance 1. Therefore, for $C' = \mathsf{long}(C)$ it follows that $\mathsf{chance}(C') = \mathsf{chance}(C)$. So,

$$\begin{aligned} \sum_{C' \in LC_k} \mathsf{chance}(C') &= \sum_{C \in LMC_k} \mathsf{chance}(\mathsf{long}(C)) \\ &= \sum_{C \in LMC_k} \mathsf{chance}(C). \end{aligned}$$

It remains for us to indicate that

$$\sum_{k=1}^{\infty} \sum_{C \in LMC_k} \mathsf{chance}(C) \leq \frac{\alpha^2}{55}.$$

Note that

$$\sum_{k=1}^{\infty} \sum_{C \in LMC_k} \mathsf{chance}(C) = \sum_{C \in LMC} \mathsf{HS}(C) \cdot \mathsf{chance}(C).$$

Thus, our interest is to prove

$$\sum_{C \in LMC} \mathsf{HS}(C) \cdot \mathsf{chance}(C) \leq \frac{\alpha^2}{55}.$$

7.3 A recurrence relation

Given $x, y \in \mathbb{Z}$ and $z \in \mathbb{N}$ with $k = 1, 2, 3, 4$, let us represent the sum of probabilities of main codes satisfying conditions $(LMC - a)$ and $(LMC - b)$ by $S_k(x, y, z)$. The values of HS and VS are equal to x and y respectively, and z terms are gotten, with the last term being k. Major codes that satisfy $(LMC - e)$ have VS equal to zero. It can be resolved from conditions $(LMC - c)$, $(LMC - d)$, and $(LMC - e)$ in equation (24) that

$$\sum_{C \in LMC} \mathsf{HS}(C) \cdot \mathsf{chance}(C) \leq \sum_{x=1}^{\infty} \sum_{y=0}^{\infty} \sum_{z=1}^{\infty} x(S_3(x, y, z) + S_4(x, y, z)). \tag{26}$$

For technical reasons, we will adjust the type 3 shift to $(-1, 1)$. This alteration does not influence the number of contours $\mathsf{tour}(\omega)$ or the number of pluses within these contours.

Considering $(LMC - a)$ of (24), the $S_k(x, y, z)$ satisfy the initial condition

$$S_k(x, y, 1) = \begin{cases} 1, & \text{if } x = 0, y = -1 \text{ and } k = 1, \\ 0, & \text{in all the other cases,} \end{cases}$$

by the $(LMC - b)$ the numbers $S_k(x,y,z)$ satisfy

$$\begin{cases} S_1(x,y,z+1) & = S_1(x,y+1,z) + S_2(x,y+1,z) \\ S_2(x,y,z+1) & = \beta[S_1(x-1,y,z) + S_2(x-1,y,z) + S_3(x-1,y,z) + S_4(x-1,y,z)] \\ S_3(x,y,z+1) & = \alpha[S_2(x+1,y-1,z) + S_3(x+1,y-1,z) + S_4(x+1,y-1,z)] \\ S_4(x,y,z+1) & = (1-\alpha)[S_2(x,y-1,z) + S_3(x,y-1,z) + S_4(x,y-1,z)] \end{cases}$$

We denote

$$S_i(z) = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} p^{-x} q^{-y} S_i(x,y,z), \qquad \text{for each } i = 1,2,3,4,$$

where p and q are positive parameters such that $x < p^{-x}$ and $1 < q^{-y}$. This way the sum in (26) is bounded superiorly by

$$\sum_{z=1}^{\infty} [S_3(z) + S_4(z)].$$

Furthermore, they satisfy the initial condition $S_1(1) = q$ and $S_2(1) = S_3(1) = S_4(1) = 0$. Likewise, we get the following recurrence conditions

$$\begin{cases} S_1(z+1) & = q[S_1(z) + S_2(z)], \\ S_2(z+1) & = \beta/p\,[S_1(z) + S_2(z) + S_3(z) + S_4(z)], \\ S_3(z+1) & = p\alpha/q\,[S_2(z) + S_3(z) + S_4(z)], \\ S_4(z+1) & = (1-\alpha)/q\,[S_2(z) + S_3(z) + S_4(z)]. \end{cases}$$

Owing to relation between $S_3(z)$ and $S_4(z)$, we will define the following quantities

$$S_1^*(z) = S_1(z), \quad S_2^*(z) = S_2(z), \quad S_3^*(z) = S_3(z) + S_4(z),$$

which has the following initial condition $S_1^*(1) = q, \quad S_2^*(1) = S_3^*(1) = 0$. Also,

$$\begin{cases} S_1^*(z+1) & = & q[S_1^*(z) + S_2^*(z)], \\ S_2^*(z+1) & = & \beta/p[S_1^*(z) + S_2^*(z) + S_3^*(z)], \\ S_3^*(z+1) & = & r[S_2^*(z) + S_3^*(z)], \end{cases}$$

where, $r = \dfrac{p\alpha + (1-\alpha)}{q}$.

Let be $S^*(z) = (S_1^*(z), S_2^*(z), S_3^*(z))$. So, we rewrite the recurrence equations by $S^*(z+1) = S^*(z) \cdot B$. Therefore, $S^*(z) = S^*(1) \cdot B^{z-1}$, where

$$B = \begin{pmatrix} q & \beta/p & 0 \\ q & \beta/p & r \\ 0 & \beta/p & r \end{pmatrix}.$$

Let us consider

$$N = \begin{pmatrix} q & \beta/p & 0 \\ q & \beta/p & r \\ q & \beta/p & r \end{pmatrix}.$$

Since $B \leq N$, then

$$S^*(z) \leq S^*(1) \cdot N^{z-1}.$$

The matrix N has three diverse eigenvalues, namely,

$$\lambda_{PF} = \frac{r+q+\beta/p+\sqrt{\Delta}}{2}, \qquad \lambda_2 = \frac{r+q+\beta/p-\sqrt{\Delta}}{2}, \qquad \lambda_3 = 0,$$

where $\Delta = (r+q+\beta/p)^2 - 4qr$ and λ_{PF} is the Perron-Frobenius eigenvalue [20]. For $\lambda_{PF} < 1$, we must possess

$$\beta < p(1+qr-q-r). \tag{27}$$

The p and q values that maximize the right side of (27) are

$$p = \frac{4\alpha - 3 + \sqrt{9-8\alpha}}{8\alpha} \quad \text{and} \quad q = \sqrt{p\alpha + 1 - \alpha}. \tag{28}$$

Note that p and q are reducing as a function of $\alpha \in (0,1)$, and for each α, $p < q < 1$.

Given p, q and α. The λ_{PF} is an increasing function pertaining to β. For the values of p and q in (28), we have $\dfrac{\alpha^2}{55} < p(1+qr-q-r)$. Thus, we have that $\lambda_{PF} < 1$ for

$$\beta < \frac{\alpha^2}{55}. \tag{29}$$

From now on, we will employ the upper bound of β on the right side of (29).

The eigenvectors linked to λ_{PF}, λ_2 and λ_3 are

$$\begin{aligned} v_{\lambda_{PF}} &= \left(v_{11}, \frac{\beta}{pq}v_{11}, \frac{r\beta}{(\lambda_{PF}-r)pq}v_{11}\right) \quad \text{with } v_{11} \in \mathbb{R}\setminus\{0\}, \\ v_{\lambda_2} &= \left(v_{21}, \frac{\beta}{pq}v_{21}, \frac{r\beta}{(\lambda_2-r)pq}v_{21}\right) \quad \text{with } v_{21} \in \mathbb{R}\setminus\{0\}, \\ v_{\lambda_3} &= (0, v_{31}, -v_{31}) \quad \text{with } v_{31} \in \mathbb{R}\setminus\{0\}. \end{aligned}$$

As well-known, we can write

$$N = V^{-1}DV,$$

where

$$D = \begin{pmatrix} \lambda_{PF} & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & a & b \\ 1 & a & c \\ 0 & 1 & -1 \end{pmatrix}$$

for $a = \beta/pq$, $b = r\beta/((\lambda_{PF}-r)pq)$ and $c = r\beta/((\lambda_2-r)pq)$. In addition,

$$V^{-1} = \begin{pmatrix} -\dfrac{a+c}{b-c} & \dfrac{a+b}{b-c} & -a \\ \dfrac{1}{b-c} & -\dfrac{1}{b-c} & 1 \\ \dfrac{1}{b-c} & -\dfrac{1}{b-c} & 0 \end{pmatrix}.$$

Let us calculate $S^*(1)N^{z-1}$, where $S^*(1) = \begin{pmatrix} q & 0 & 0 \end{pmatrix}$.

$$\begin{aligned} S^*(1)N^k &= S^*(1)V^{-1}D^kV \\ &= q\begin{pmatrix} -\dfrac{a+c}{b-c} & \dfrac{a+b}{b-c} & -a \end{pmatrix} \begin{pmatrix} \lambda_{PF}^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 1 & a & c \\ 0 & 1 & -1 \end{pmatrix} \\ &= q\begin{pmatrix} -\dfrac{a+c}{b-c}\lambda_{PF}^k + \dfrac{a+b}{b-c}\lambda_2^k & -\dfrac{a+c}{b-c}a\lambda_{PF}^k + \dfrac{a+b}{b-c}a\lambda_2^k & -\dfrac{a+c}{b-c}b\lambda_{PF}^k + \dfrac{a+b}{b-c}c\lambda_2^k \end{pmatrix} \end{aligned}$$

Using $k = z - 1$,

$$S_3^*(z) \le q\left(-\frac{a+c}{b-c} b\lambda_{PF}^{z-1} + \frac{a+b}{b-c} c\lambda_2^{z-1}\right).$$

As $\lambda_{PF} < 1$, we got from the Perron-Frobenius theorem that $|\lambda_2| < 1$. So,

$$\begin{aligned}\sum_{z=1}^{\infty} S_3^*(z) &\le -\frac{a+c}{b-c} bq \sum_{z=0}^{\infty} \lambda_{PF}^z + \frac{a+b}{b-c} cq \sum_{z=0}^{\infty} \lambda_2^z \\ &= -\frac{a+c}{b-c} bq \frac{1}{1-\lambda_{PF}} + \frac{a+b}{b-c} cq \frac{1}{1-\lambda_2}.\end{aligned}$$

As

$$a+b = \frac{\beta\lambda_{PF}}{pq(\lambda_{PF}-r)}, \quad a+c = \frac{\beta\lambda_2}{pq(\lambda_2-r)}, \quad b-c = \frac{r\beta(\lambda_2-\lambda_{PF})}{pq(\lambda_{PF}-r)(\lambda_2-r)},$$

it follows,

$$-\frac{a+c}{b-c} bq = -\frac{\beta\lambda_2}{p(\lambda_2-\lambda_{PF})} \quad \text{and} \quad \frac{a+b}{b-c} cq = \frac{\beta\lambda_{PF}}{p(\lambda_2-\lambda_{PF})}.$$

Thus,

$$\begin{aligned}\sum_{z=1}^{\infty} S_3^*(z) &\le -\frac{\beta\lambda_2}{p(\lambda_2-\lambda_{PF})(1-\lambda_{PF})} + \frac{\beta\lambda_{PF}}{p(\lambda_2-\lambda_{PF})(1-\lambda_2)} \\ &\le \frac{\beta}{p(\lambda_{PF}-\lambda_2)}\left[\frac{\lambda_{PF}}{1-\lambda_{PF}} - \frac{\lambda_{PF}}{1-\lambda_2}\right] \\ &= \frac{\beta\lambda_{PF}}{p(1-\lambda_{PF})(1-\lambda_2)}. \qquad (30)\end{aligned}$$

The expression (30) will be denoted by $UP(p, q, \alpha, \beta)$.
Proof of item (A. 1) of Theorem 1. Note that, for $\beta \in (0, \alpha^2/55)$, $UP(p, q, \alpha, \beta)$ is growing as a function of β. So, if $UP(p, q, \alpha, \alpha^2/55)$ is less than 1, then $UP(p, q, \alpha, \beta)$ will be less than 1. As $(1-\lambda_{PF})(1-\lambda_2) = 1 + qr - q - r - \beta/p$,

$$UP(p, q, \alpha, \beta) = \frac{\beta\lambda_{PF}}{p(1+qr-q-r)-\beta}.$$

Therefore,

$$UP(p, q, \alpha, \beta) < 1 \iff \beta(\lambda_{PF}+1) < p(1+qr-q-r). \qquad (31)$$

Supposing $p = (4\alpha - 3 + \sqrt{9-8\alpha})/8\alpha$, $q = \sqrt{p\alpha + 1 - \alpha}$ and $r = (p\alpha + 1 - \alpha)/q$, one can prove

$$p(1+qr-q-r) \ge \frac{\alpha^2}{27}.$$

So, when $\beta = \alpha^2/55$ we have $2\beta < p(1+qr-q-r)$. Once $0 < \lambda_{PF} < 1$,

$$\beta(\lambda_{PF}+1) < p(1+qr-q-r).$$

Thus,

$$UP\left(\frac{4\alpha-3+\sqrt{9-8\alpha}}{8\alpha}, \sqrt{p\alpha+1-\alpha}, \alpha, \frac{\alpha^2}{55}\right) < 1.$$

□

At this point, we will introduce some definitions and outcomes which was previously described in [8, 9].

For words $W = (a_1, \ldots, a_m)$ and $V = (b_1, \ldots, b_n)$, whose $|W| \leq |V|$, we say *positions* of W in V to the integer value in the interval $[0, n-m]$. The word W *enters* word V at the position k if

$$\forall \ \ i \in \mathbb{Z} : 1 \leq i \leq m \Rightarrow a_i = b_{i+k}.$$

We say that a word W is *self-overlapping* case there is a word V such that $|V| < 2 \cdot |W|$ and word W enters word V at two different positions. Therefore, we say that a word is *self-avoiding* case it is not self-overlapping.

Now, we shall elucidate as a *substitution operator* acts on $\mathcal{M}$. Lets $\rho \in [0, 1]$ be. Fixed word G and word H, where G is self-avoiding, a generic substitution operator, informally speaking, turns every entrance of the word G in a long word into a word H with probability ρ or does not alter with a probability $1 - \rho$ independently of states of all the other components.

Let $\mathsf{P}_1, \ldots, \mathsf{P}_j$ be a finite sequence of substitution operators and we assume $\mathsf{P} = \mathsf{P}_1 \circ \cdots \circ \mathsf{P}_j$. We represent the *generalized discrete substitution process* by the sequence of measures $\nu, \nu\mathsf{P}, \nu\mathsf{P}^2, \ldots, \nu\mathsf{P}^t, \ldots$, where P^t signifies the t-th composition of operator P. So,

$$\nu_t(W) = \nu\mathsf{P}^t(W) \qquad \text{for every word } W.$$

Toom *et. al.* **(2011).** *Let us consider a generalized discrete substitution approach $\nu_n = \nu\mathsf{P}^n$, where $\mathsf{P} = \mathsf{P}_1\mathsf{P}_2 \cdots \mathsf{P}_j$ as in the above definition. Let $R \subset \mathbb{A}^{\mathbb{Z}}$ be some subset of σ-algebra $\mathbb{A}^{\mathbb{Z}}$. Then, if $\nu_n(c) \leq \delta$ (respectively $\nu_n(c) \geq \epsilon$) for all $c \in R$, then P has an invariant measure μ such that $\mu(c) \leq \delta$ (respectively $\mu(c) \geq \epsilon$) for every $c \in R$, where $\delta, \epsilon > 0$ are positive constants.*

Proof of item (B. 1) of Theorem 1. We represent cylinder $C = \{x \in \mathcal{A}^{\mathbb{Z}} : x_{i_0} = \oplus \text{ for some } i_0 \in \mathbb{Z}\}$ and $\delta = UP(p, q, \alpha, \beta)$, where $p = (4\alpha - 3 + \sqrt{9 - 8\alpha})/8\alpha$, $q = \sqrt{p\alpha + 1 - \alpha}$ and $\beta \in (0, \alpha^2/55)$. Through (31),

$$\mu_t(\oplus) = \delta_{\ominus}(\mathsf{F}_\beta\mathsf{C}_\alpha)^t(\oplus) \leq UP(p, q, \alpha, \beta), \qquad \text{for all } t \in \mathbb{Z}_+.$$

So, by **Toom et. al., (2011)**, the operator $\mathsf{F}_\beta\mathsf{C}_\alpha$ has a invariant measure ν. such that

$$\nu(\oplus) \leq UP(p, q, \beta, \alpha) < 1.$$

□

8 Open problems

Here, we introduce some questions motivated by our research. We have demonstrated that μ_t exhibits the behaviors of ergodicity and non-ergodicity. Nevertheless, the lack of a certain kind of monotonicity does not enable us to affirm the operator behavior, i.e.

Problem 1. *There is $B \subset (0, 1)$ such that, for α, $\beta \in B$ if $\lim_{t\to\infty} \mu_t(\oplus) = 1$, then $\lim_{t\to\infty} \nu(\mathsf{F}_\beta\mathsf{C}_\alpha)(\oplus) = 1$, for all $\nu \in \mathcal{M}$.*

We suspect that our operators have a certain kind of monotonicity, in the following sense

Problem 2. *Let $\alpha \in (0,1]$ be fixed and values β_1 and β_2 belong to $(0,1)$. If $\beta_1 < \beta_2$, then*

$$\delta_\ominus(\mathsf{F}_{\beta_1}\mathsf{C}_\alpha)^t(\oplus) < \delta_\ominus(\mathsf{F}_{\beta_2}\mathsf{C}_\alpha)^t(\oplus) \text{ for each } t \in \mathbb{N}.$$

A more general scenario where particles alter their stimulus, turning a positive stimulus into a negative stimulus or turning a negative stimulus into a positive, is a case to be considered. It is summarized as follows

Problem 3. *Prove analogs of Theorems 1 and 2 for the process*

$$\delta_\ominus(\mathsf{F}_\beta^\gamma\mathsf{C}_\alpha)^t,$$

in which with an independent way plus changes to minus with probability β and minus becomes plus with probability γ.

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