Dynamic Aspects of the Flip-Annihilation Process

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Abstract

An one-dimensional interacting particle system is revisited. It has discrete time and its components are located in the integers set. These components can disappear in the functioning process. Each component assumes two possible states, called *plus* and *minuses* and interact at every time step only with their nearest neighbors. The following two transformations happen: The first one is called *flip*, under its action a component in state minus turns into a plus with probability β. The second one is called *annihilation*, under its action, whenever a component in state plus is a left neighbor of a component in state minus, both components disappear with probability α. Let us consider a set of initial measure to the process. For these measures, we show the upper bound for the mean time of convergence, which is a function of the initial measure. Moreover, we obtain upper bound to the mean quantity of minuses on the process in each time step. Considering the initial measure concentrated at the configuration whose components are in the state minuses, we improved a well-known result, that the process is non-ergodic when $\beta < \alpha^2/250$. Now, we are able to offer non-ergodicity when $\beta < 9\alpha^2/1000$. Also, we established new conditions to the ergodicity of the process. Finally, we performed some Monte Carlo simulations for this process.

1 Introduction and Main Results

From the mid-twentieth century onwards, the development of a new part of stochastic process theory, called the local interaction theory of stochastic processes, has begun to be developed, which is now better known as *interacting particles systems* [6, 2]. New kinds and models of interacting particle systems have appeared. These systems do not quite fit into the standard theory [2, 6, 13]. For example, the assumption that the set of components does not change in the process of interaction is considered. This assumption is not the only possible one. Here we investigate a process belonging to a class of interacting

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particle systems whose the set of components themselves appear or disappear during the process of functioning. We call such process *variable length* [13, 14, 19, 20].

The aims in this study is to provide theoretical understanding of this new class through the development of analytical techniques and computational modeling. This step precedes applications to real situations. Another study of similar processes is presented in Malyshev's works [8, 9], which are motivated by some connections between computer science and quantum gravity.

Our process, which we will call Flip-Annihilation, was proposed in [19] and studied analytically and computationally in [11, 12, 13, 20]. This process has a discrete time, whose components are located in $\mathbb{Z}$. Each component assumes two states, called plus and minus and denoted by $\oplus$ and $\ominus$, respectively. Informally speaking, in each step of time two transformations occur:

- The first called *flip* and denoted by F_β transform any minus into plus with probability β independently from each other.
- The second called *annihilation* and denoted by A_α is variable length and under its action whenever a plus is the left neighbor of a minus, both disappear with probability α, independently from other occurrences of this type.

As in [19], we denote

$$\mu_t = \delta_\ominus(\mathsf{F}_\beta \mathsf{A}_\alpha)^t, \tag{1}$$

where $\delta_\ominus$ is the measure concentrated in the configuration all minus, whose all particles assume the state minus.

In this work, we define a class of measures, which are in some sense "close" of an invariant distribution of our process and we verify: the upper bound for the mean time that it takes to reach the invariant distribution also we verify in each step of time t, the upper bound of the mean number of components in the state minus in the system. In both cases, these upper bounds are functions of the initial distribution.

The Flip-Annihilation shows a kind of phase transition [19]. In the parameter space there are two regions: at one, (1) is ergodic; at another one (1) is non-ergodic. In this work we expanded this non-ergodic region. Our result put us closer to the "critical curve" case there is.

In our class, we called *alphabet* any non-empty finite set $\mathcal{A}$, and call its elements *letters*. We call a *word* in the alphabet $\mathcal{A}$ any finite sequence of terms, everyone of which is an element of $\mathcal{A}$. The length of a word W is the number of letters in it and is denoted by $|W|$. Any letter may be treated as a word of length one. There is the empty word, denoted by Λ, whose length is zero. Let us call the *dictionary* and denote by $dic(\mathcal{A})$ the set of words in the alphabet $\mathcal{A}$. We denote by $\mathbb{Z}$ the set of integer numbers and $\mathcal{A}^\mathbb{Z}$ the set of bi-infinite sequences, whose terms are elements of $\mathcal{A}$.

Let us denote by $\mathbb{A}$ the discrete topology on $\mathcal{A}$. We consider probability measure on the σ-algebra $\mathbb{A}^\mathbb{Z}$ on the product space $\mathcal{A}^\mathbb{Z}$ endowed with the topology - product of discrete topologies on all the copies of $\mathcal{A}$. Since $\mathcal{A}$ is finite, it is compact in the discrete topology, and by Tychonoff's compact theorem [10], $\mathcal{A}$ also is compact.

In the usual way we define translation on $\mathbb{Z}$, then on $\mathcal{A}^\mathbb{Z}$, then on the set of normed measures on $\mathcal{A}^\mathbb{Z}$ and call a measure *uniform* if it is invariant under all translations. This is, for all word $W = (a_1, \ldots, a_n)$,

$$\mu(W) = \mu(a_1, \ldots, a_n) = \mu(s_{i+1} = a_1, \ldots, s_{i+n} = a_n),$$

for all $i \in \mathbb{Z}$. We denote by $\mathcal{M}_{\mathcal{A}}$ the set of uniform probability measures in $\mathcal{A}^{\mathbb{Z}}$.

At this point we will bring some definitions and results described in [20, 14].

Given two words $W = (a_1, \ldots, a_m)$ and $V = (b_1, \ldots, b_n)$, where $|W| \leq |V|$, we call the integer number in the interval $[0, n-m]$ *positions* of W in V. We say that W *enters* V at a position k if

$$\forall \;\; i \in \mathbb{Z} : 1 \leq i \leq m \Rightarrow a_i = b_{i+k}.$$

We call a word W *self-overlapping* if there is a word V such that $|V| < 2 \cdot |W|$ and W enters V at two different positions. A word is called *self-avoiding* if it is not self-overlapping. In particular, the empty word, every word consisting of one letter and every word consisting of two different letters are self-avoiding.

A generic *substitution operator* acts from $\mathcal{M}_{\mathcal{A}}$ to $\mathcal{M}_{\mathcal{A}}$ as follows: given two words G and H, where G is self-avoiding, and a real number $\rho \in [0, 1]$, a generic substitution operator, informally speaking, substitutes every entrance of the word G in a long word by the word H with a probability ρ or leaves it unchanged with a probability $1 - \rho$ independently of states of all the other components. We denote this operator by $(G \xrightarrow{\rho} H)$.

Let $\nu \in \mathcal{M}$ and $\mathsf{P} = \mathsf{P}_1 \circ \cdots \circ \mathsf{P}_j$, where $\mathsf{P}_1, \ldots, \mathsf{P}_j$ be a finite sequence of substitution operator. Then we define the *generalized discrete substitution process* ν_t, where $\nu_0 = \nu$, as follows:

$$\nu_t(W) = \nu \mathsf{P}^t(W) \qquad \text{for every word } W.$$

and P^t denotes the t-th composition of operator P.

Toom et. al., (2011). *Let us consider a generalized discrete substitution process $\nu_n = \nu\mathsf{P}^n$, where $\mathsf{P} = \mathsf{P}_1\mathsf{P}_2 \cdots \mathsf{P}_j$ as in above definition. Let $R \subset \mathbb{A}^{\mathbb{Z}}$ be some subset of σ-algebra $\mathbb{A}^{\mathbb{Z}}$. Then, if $\nu_n(c) \leq \delta$ (respectively $\nu_n(c) \geq \epsilon$) for all $c \in R$, then P has an invariant measure μ such that $\mu(c) \leq \delta$ (respectively $\mu(c) \geq \epsilon$) for every $c \in R$, where $\delta, \epsilon > 0$ are positive constants.*

Lets $\mathcal{A}^{\mathbb{Z}} = \{\ominus, \oplus\}^{\mathbb{Z}}$ and $x, y \in \mathcal{A}^{\mathbb{Z}}$. We say that two configurations x and y are *close* to each other if the set $\{i \in \mathbb{Z} : x_i \neq y_i\}$ is finite. A configuration is called a *island of minus* if it is close to the configuration "all plus", and we denote the set of island of minus by Δ. Given $x \in \Delta$, we define the *population* of x, and denote by $\mathsf{Pop}(x)$, as the quantity of minus in the island x, this is,

$$\mathsf{Pop}(x) = \#\{i \in \mathbb{Z} : x_i = \ominus\},$$

where $\#(\cdot)$ denotes the cardinality of the set.

We denote by $\mathcal{A}_{\ominus}$ the set of normalized measures in the countable set, Δ. A measure belonging to $\mathcal{A}_{\ominus}$ is called an *archipelago of minus*. From now on, case do not stated, μ denotes an archipelago of minus.

The operators $\mathsf{F}_\beta : \mathcal{A}_{\ominus} \to \mathcal{A}_{\ominus}$ and $\mathsf{A}_\alpha : \mathcal{A}_{\ominus} \to \mathcal{A}_{\ominus}$ are well defined. Thus, if μ is fixed, it is possible to define the random variable

$$\tau_\mu = \inf\left\{t \geq 0 : \mu(\mathsf{F}_\beta\mathsf{A}_\alpha)^t = \delta_{\oplus}\right\}. \tag{2}$$

The infimum of the empty set is ∞. The random varianle τ_μ denotes the time to attain the configuration "all plus" having $\mathsf{F}_\beta\mathsf{A}_\alpha$, started on μ.

Note that $\mu = \sum_{i=1}^{\infty} k_i \delta_{x^i}$, where $k_1 > 0, k_2 > 0, \ldots; k_1 + k_2 + \cdots = 1$ and $x^i \in \Delta$ for $i \in \mathbb{N}$. We define the *maximum population* of μ as

$$\mathsf{M}(\mu) = \max\left\{\mathsf{Pop}(x^i) : \mu = \sum_{i=1}^{\infty} k_i \delta_{x^i}\right\}.$$

If there is no such maximum, we say that $\mathsf{M}(\mu) = \infty$.

Generally, when we talk about probabilistic cellular automata, a process P means a sequences of measure

$$\mu, \mu\mathsf{P}, \mu\mathsf{P}^2, \ldots, \mu\mathsf{P}^t, \ldots$$

However, we can define this process through a sequence

$$x^0, x^1, x^2, \ldots, x^t, \ldots \tag{3}$$

where x^0 has distribuition μ and x^t has distribuition $\mu\mathsf{P}^t$. This definition is intuitive and can be found in more detail in [7].

For $x \in \Delta$ consider (3) to describe the process. Lets δ_x the distribution concentrated in x and $\delta_x(\mathsf{F}_\beta\mathsf{A}_\alpha)^t$ the distribution of x^t, we define the *population of x in the time t* by $\mathsf{Pop}(x^t)$. This random variable represents the quantity of minus on the island x on the t-th time step. Given $\mu \in \mathcal{A}_\ominus$, we define the *maximum population of μ in the time t* by

$$\mathsf{M}(\mu, t) = \max\left\{\mathsf{Pop}\left((x^i)^t\right) : \mu\mathsf{P}^t = \sum_{i=1}^{\infty} k_i \delta_{(x^i)^t}\right\},$$

where $(x^i)^t$ denotes the i-th island in time t and if there is no such maximum, we say that $\mathsf{M}(\mu, t) = \infty$.

It is clear for a δ-measure concentrated in a configuration $x \in \Delta$: if $\alpha > 0$ or $\beta > 0$ we have that $\delta_x(\mathsf{F}_\beta\mathsf{A}_\alpha)^t$ tends to $\delta_\oplus$ when t tends to infinity. However, it is not clear how fast is this convergence and how this quantity of minus behaves in the evolution of the system. Theorems 1 and 2 provide us information in this direction.

Theorem 1. *Given $\mu \in \mathcal{A}_\ominus$ and τ_μ as defined in (2). If $\mathsf{M}(\mu)$ is finite then*

$$\mathbb{E}(\tau_\mu) \leq \begin{cases} f_\mu(\alpha), & \text{if } \alpha \geq \dfrac{\mathsf{M}(\mu)}{g_\mu(\beta)}, \\ g_\mu(\beta), & \text{if } \alpha < \dfrac{\mathsf{M}(\mu)}{g_\mu(\beta)} \end{cases}$$

where $f_\mu(\alpha) = \dfrac{\mathsf{M}(\mu)}{\alpha}$ *and* $g_\mu(\beta) = \displaystyle\sum_{i=1}^{\mathsf{M}(\mu)} \frac{1}{1-(1-\beta)^i}$.

Theorem 2. *Given natural value t and $\mu \in \mathcal{A}_\ominus$. If $\mathsf{M}(\mu, t)$ is finite then*

$$\mathbb{E}\left(\mathsf{M}(\mu, t)\right) \leq (1-\beta)^{t-1}\left(\mathsf{M}(\mu) + \alpha\beta^{\mathsf{M}(\mu)} - \alpha\right).$$

We denote the Theorems 1, 2 and 3 in [19] respectively by **(r1)**, **(r2')** and **(r3)**. The results ***(r1)*** *If $2\beta > \alpha$, the measures μ_t tend to $\delta_\oplus$ when $t \to \infty$ and* ***(r2')*** *For all natural t the frequency of pluses in the measure μ_t does not exceed* $300 \cdot \beta/\alpha^2$ shows that there is some kind of phase transition for (1). In [11], the **(r2')** was improved; in fact, it allows us to substitute 250 instead of 300. The **(r2')** after this modification is denoted by **(r2)**.

About ergodicity of our operator, it was shown: ***(r3)*** *Take any $\mu \in \mathcal{M}_{\{\ominus,\oplus\}}$ and suppose that $\beta > 0$ and $\mu(\ominus) \in \left(0, \dfrac{1}{2(1-\beta)}\right)$. Then the measures $\mu(\mathsf{F}_\beta\mathsf{A}_\alpha)^t$ tend to $\delta_\oplus$ when $t \to \infty$.*

Theorem 3 give us a new condition for which $\mu(\mathsf{F}_\beta\mathsf{A}_\alpha)^t$ converges to $\delta_\oplus$, and Theorem 4 give us a region where μ_t is non ergodic. In fact, contains the region of non-ergodicity described in **(r2)**.

Theorem 3. *Given $\mu \in \mathcal{M}_{\{\ominus,\oplus\}}$. If $\mu(\ominus) \in \left(0, \frac{\beta}{2\alpha(1-\beta)}\right)$, then $\mu(\mathsf{F}_\beta \mathsf{A}_\alpha)^t \to \delta_\oplus$, when $t \to \infty$.*

Theorem 4. *For $\alpha \in (0,1)$. If $\beta < \frac{9\alpha^2}{1000}$ then*

(A. 4) for all natural value t, $\mu_t(\oplus) < 1$;

(B. 4) there is a measure, ν, such that $\nu(\mathsf{F}_\beta \mathsf{A}_\alpha) = \nu$, where $\nu(\oplus) < 1$.

2 Two auxiliary operators

Here, we introduce the operators Neutralization and Elimination.

Given $\Omega_\odot = \{\oplus, \ominus, \odot\}^{\mathbb{Z}}$, where $\odot$ is called *dot*. We denote by $\Delta_\odot$ the set containing all $x \in \Omega_\odot$ such that $\mathsf{Pop}(x)$ is finite. Thus, there is a copy of Δ in $\Delta_\odot$, this fact will be used by us. Note that, if $x \in \Delta_\odot$, then $\mathsf{Pop}(x)$ is finite, however can exist infinite positions in the states plus or dot. That is, if $x \in \Delta_\odot$, then there are indices i_0, j_0, with $i_0 < j_0$ such that $x_{i_0} = x_{j_0} = \ominus$ and $x_k \neq \ominus$ for all $k < i_0$ and $k > j_0$. Now, we define by $\mathcal{A}_\odot$ the set of normalized measures in the countable set, $\Delta_\odot$. The symbol $\odot^n$ denotes a word with n consecutive letters equal to $\odot$. In particular, we denote $\oplus\ominus$ by $\oplus \odot^0 \ominus$. Let us consider an operator, which we will call *neutralization*, $\mathsf{N}_\alpha : \mathcal{A}_\odot \to \mathcal{A}_\odot$, which acts as follows: every word $\oplus \odot^n \ominus$ is transformed into a word $\odot^{n+2}$ with probability α, or remains unchanged with probability $1-\alpha$, for all $n \in \mathbb{N}$. N_α is not variable length operator. It can be shown that the operator N_α is linear.

If a particle goes to the state $\odot$, it remains in this state forever.

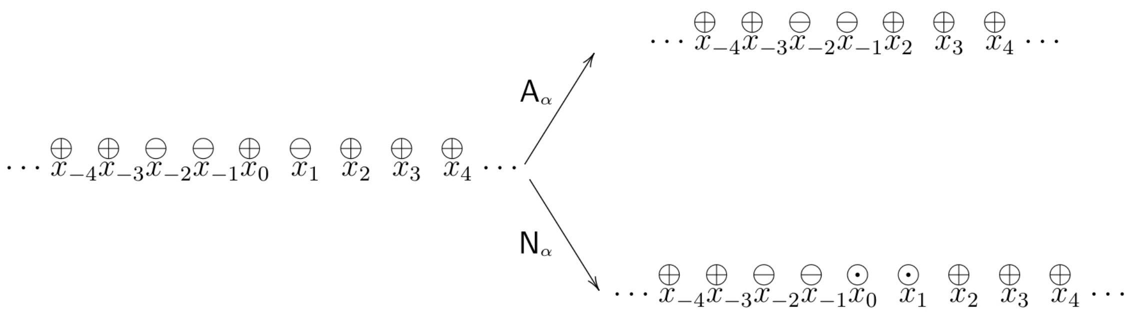


Figure 1: Illustration of the action of operators N_α and A_α on an island whose $\mathsf{Pop}(x) = 3$.

In Figure 2, we illustrate a possible action (it has a positive probability of occuring) of the operators annihilation and neutralization in a fragment of an island $x \in \Delta$. The operator annihilation eliminate the positions. On the other hand, the operator neutralization does not eliminates the positions, it acts changing the state of the components.

Given $x \in \Delta$ and δ_x its respective normalized measure. By the definitions of N_α and A_α, we have

$$\delta_x \mathsf{A}_\alpha(\ominus) = \frac{\delta_x \mathsf{N}_\alpha(\ominus)}{1 - \delta_x \mathsf{N}_\alpha(\odot)}.$$

So, for each $t \in \mathbb{N}$

$$\delta_x \mathsf{A}^t_\alpha(\ominus) = \frac{\delta_x \mathsf{N}^t_\alpha(\ominus)}{1 - \delta_x \mathsf{N}^t_\alpha(\odot)}. \tag{4}$$

Now we will describe the operator *elimination* $\mathsf{E}_\alpha : \mathcal{A}_\odot \to \mathcal{A}_\odot$, this is a particular case of the operator neutralization. Informally speaking, given $x \in \Delta$ (that is, its copy in $\Delta_\odot$) the operator elimination acts on the minus leftmost and transforms the word $\oplus \odot^n \ominus$ into $\odot^{n+2}$ with probability α or it remains the same with probability $1-\alpha$. The operator elimination transforms only one occurrence of the word $\oplus \odot^n \ominus$ into $\odot^{n+2}$. Since E_α is a particular case of the operator neutralization and this is linear. Thus E_α is also linear. In probabilistic cellular automata the standard operators are linear, and once that the operators F_β, N_α and E_α are in this context. We concluded their linearity.

Lemma 1. *Given $\mu \in \mathcal{A}_\ominus$ we have*

$$\mu \mathsf{A}_\alpha(\ominus) \leq \frac{\mu \mathsf{E}_\alpha(\ominus)}{1-\mu \mathsf{E}_\alpha(\odot)}.$$

Proof. We know that there is a copy of $\mathcal{A}_\ominus$ in $\mathcal{A}_\odot$, so $\mu\mathsf{E}_\alpha$ for all $\mu \in \mathcal{A}_\ominus$, makes sense. Moreover, given $\mu \in \mathcal{A}_\ominus$ we can write μ as a convex combination of δ_{x^i} measures where $x^i \in \Delta$ for all $i \in \mathbb{N}$. Consequently, from (4) it follows that

$$\mu \mathsf{A}_\alpha(\ominus) = \frac{\mu \mathsf{N}_\alpha(\ominus)}{1-\mu \mathsf{N}_\alpha(\odot)}.$$

Since E_α is a particular case of N_α, in which only one occurrence of the word $\oplus \odot^n \ominus$ is transformed into $\odot^{n+2}$, for some $n \in \mathbb{N}$, with probability α, it follows that

$$\frac{\mu \mathsf{N}_\alpha(\ominus)}{1-\mu \mathsf{N}_\alpha(\odot)} \leq \frac{\mu \mathsf{E}_\alpha(\ominus)}{1-\mu \mathsf{E}_\alpha(\odot)}.$$

□

3 The operator $\mathsf{F}_\beta \mathsf{E}_\alpha$ and the process Z

Initially, we will describe a correspondence between the operator $\mathsf{F}_\beta\mathsf{E}_\alpha$ acting on δ-measures, δ_x, where x belongs to the copy of Δ in $\Delta_\odot$ and the process Z.

- About operator F_β, which transforms minus into plus with probability β independently of what happen in the other positions. This is analogous with the first half step of the process Z.

- About operator E_α, which transforms a single component in the state minus into state point with probability α. This is analogous with the second half step in the process Z.

Thus, we associate the operator $\mathsf{F}_\beta\mathsf{E}_\alpha$ (acting first F_β and then E_α in this order) with the action of the first half step and second half step of process Z.

The operators F_β and E_α are linear. Hence $\mathsf{F}_\beta\mathsf{E}_\alpha$ is linear. Therefore, given μ belonging to a copy of $\mathcal{A}_\ominus$ in $\mathcal{A}_\odot$, μ is a convex combination of δ-measures. So, once we describe $\mathsf{F}_\beta\mathsf{E}_\alpha$ acting in δ-measures, we define $\mu\mathsf{F}_\beta\mathsf{E}_\alpha$.

For $x \in \Delta_\odot$ we have $Z_0 = \mathsf{Pop}(x)$ and Z_t matches the quantity of minus on the island x in time t, x^t, which has distribution $\delta_x(\mathsf{F}_\beta\mathsf{E}_\alpha)^t$. Note that when $\mu = \sum_{i=1}^{\infty} k_i \delta_{x^i}$ we have

$$\mathbb{P}\left(Z_0 = \mathsf{Pop}(x^i)\right) = k_i.$$

4 Proof of Theorems 1 and 2

Let us consider two Markov Chains, whose state set is $\mathbb{Z}_+$. We will denote its transition probability from a state i to a state j by p_{ij}. For simplicity, sometimes we will use the notation M.C., instead of Markov Chain.

Our first Markov Chain will be illustrated from the following scenario: there is an urn with n balls in the initial moment. In each moment each one of the balls can be removed with probability β or remain with probability $1-\beta$, independently from each other. Its transition probabilities

$$p_{mn} = \begin{cases} 0, & \text{if} \quad m < n; \\ \binom{m}{n}(1-\beta)^n \beta^{m-n}, & \text{if} \quad m \geq n, \end{cases} \tag{5}$$

where $\beta \in [0,1]$.

Our second Markov Chain has transition probabilities

$$\begin{aligned} &p_{00} = 1, \\ &p_{nn-1} = q_n, \quad p_{nn} = 1 - q_n, \quad p_{nj} = 0 \text{ if } j \notin \{n, n-1\}, \end{aligned} \tag{6}$$

with $q_n \in [0,1]$.

Considering (5) and (6), if $q_n \in (0,1)$ and $\beta \in (0,1)$, then the absorption probability is one. Given the M. C. $(X_t)_{t\in\mathbb{N}}$ we define the random variable

$$H_n = \inf\{t \in \mathbb{N} : X_t = 0 \text{ and } X_0 = n\}.$$

By simplicity, we will denote $\mathsf{k}_n = \mathbb{E}(H_n)$. From now on, we will denote by $X = (X_t)_{t\in\mathbb{Z}_+}$ and $Y = (Y_t)_{t\in\mathbb{Z}_+}$ the Markov Chains given by (5) and (6) respectively.

For Y

$$\mathsf{k}_n^Y = \sum_{i=1}^{n} \frac{1}{q_i} \qquad \text{for } n \geq 1. \tag{7}$$

and for X

$$\mathsf{k}_n^X = \sum_{i=0}^{\infty} [1 - (1 - (1-\beta)^i)^n]. \tag{8}$$

As far as we know, the expression (8) can not be represented by a simple elementary form.

The equation (7) with $q_i = 1 - (1-\beta)^i$ is

$$\mathsf{k}_n^Y = \sum_{i=1}^{n} \frac{1}{1 - (1-\beta)^i}. \tag{9}$$

Using a coupling on the Markov chains X and Y, we are able to prove: if $X_0 = Y_0 = n$, then $\mathsf{P}(X_t \leq Y_t) = 1$ for all $t \in \mathbb{Z}_+$. Thus, we conclude that $\mathsf{k}_n^X \leq \mathsf{k}_n^Y$. More precisely,

$$\mathsf{k}_n^X = \sum_{i=0}^{\infty} [1 - (1 - (1-\beta)^i)^n] \leq \sum_{i=1}^{n} \frac{1}{1 - (1-\beta)^i} = \mathsf{k}_n^Y. \tag{10}$$

We will utilize the right side of (10) as an upper bound to the absorption mean time of X.

We will define a M. C., as a kind of composition between $(X_t)_{t\in\mathbb{Z}_+}$ and $(Y_t)_{t\in\mathbb{Z}_+}$. The following scenario describes what happens in a time step: at the first half step our urn has n balls, each ball will be removed with probability β or it will remain with probability $1-\alpha$; in the second half step, we take one and only one of the balls that remained in the urn, it will be removed with probability α or it will remain with probability $1-\alpha$.

After completing the second half step, we have completed one time step in this new M. C., which we will denote $Z = (Z_t)_{t\in\mathbb{Z}_+}$.

Let us consider two sequences of independent random variables $(a_i^t)_{i,t\in\mathbb{N}}$ and $(c^t)_{t\in\mathbb{N}}$, where

$$a_i^t = \begin{cases} 0, & \text{with probability } 1-\beta; \\ -1, & \text{with probability } \beta. \end{cases} \quad \text{and} \quad c^t = \begin{cases} 0, & \text{with probability } 1-\alpha; \\ -1, & \text{with probability } \alpha. \end{cases}$$

Note that Z_t describes the balls number remains in the urn at time t given that $Z_0 = n$. The expression (11) give us an inductive representation of Z_t,

$$Z_{t+1} = Z_t + \sum_{i=1}^{Z_t} a_i^{t+1} + c^{t+1}\chi_{\{\Psi_t>0\}}\left(\Psi_t\right), \tag{11}$$

where $\chi_{\{\cdot\}}(\cdot)$ is the indicator function and $\Psi_t = Z_t + \sum_{i=1}^{Z_t} a_i^{t+1}$. Our goal is to calculate the mean number of balls in time $t+1$ given that we start with n balls.

Proposition 1. *Let us consider the process Z and its inductive definition in* (11)*, so*

$$\mathbb{E}(Z_{t+1}) = (1-\beta)\mathbb{E}(Z_t) + \alpha[\mathbb{E}(\beta^{Z_t}) - 1]. \tag{12}$$

Proof. From (11)

$$\begin{aligned} \mathbb{E}(Z_{t+1}) &= \sum_{k=0}^{\infty} \mathbb{E}(Z_{t+1}|Z_t = k)\mathbb{P}(Z_t = k) \\ &= \sum_{k=0}^{\infty} \left[\mathbb{E}\left(k + \sum_{i=1}^{k} a_i^{t+1}\right)\mathbb{P}(Z_t = k) + \mathbb{E}\left(c^{t+1}\chi_{\{\Psi_t>0\}}\left(\Psi_t\right)\right)\mathbb{P}(Z_t = k)\right]. \end{aligned}$$

Thus,

$$\mathbb{E}(Z_{t+1}) = (1-\beta)\sum_{k=0}^{\infty} k\mathbb{P}(Z_t = k) - \alpha\sum_{k=0}^{\infty}\mathbb{P}\left(-\sum_{i=1}^{k} a_i^{t+1} < k\right)\mathbb{P}(Z_t = k).$$

Using that

$$\mathbb{P}\left(-\sum_{i=1}^{k} a_i^{t+1} < k\right) = 1 - \beta^k.$$

We conclude

$$\mathbb{E}(Z_{t+1}) = (1-\beta)\mathbb{E}(Z_t) + \alpha\left[\sum_{k=0}^{\infty}\beta^k\mathbb{P}(Z_t = k) - 1\right].$$

Since $\mathbb{E}(\beta^{Z_t}) = \sum_{k=0}^{\infty}\beta^k\mathbb{P}(Z_t = k)$, the proof of this proposition is concluded. □

Using (12), and by induction in t, we get

$$\mathbb{E}(Z_{t+1}) = (1-\beta)^{t+1}Z_0 + \alpha\sum_{i=0}^{t}(1-\beta)^i(\mathbb{E}(\beta^{Z_{t-i}}) - 1). \tag{13}$$

Since $\mathbb{E}(\beta^{Z_t}) - 1 \leq 0$ for all $t \in \mathbb{N}$ and considering only when $i = t$ on the right side of (13) we get

$$\mathbb{E}(Z_{t+1}) \leq (1-\beta)^{t+1} Z_0 + \alpha(1-\beta)^t(\mathbb{E}(\beta^{Z_0}) - 1) \leq (1-\beta)^t(Z_0 + \alpha\beta^{Z_0} - \alpha). \tag{14}$$

Proof of Theorem 1.

We know that

$$\tau_\mu = \inf\{t \geq 0 : \mu(\mathsf{F}_\beta \mathsf{A}_\alpha)^t = \delta_\oplus\} = \inf\{t \geq 0 : \mu(\mathsf{F}_\beta \mathsf{A}_\alpha)^t(\ominus) = 0\}.$$

There is a copy of $\mathcal{A}_\ominus$ in $\mathcal{A}_\odot$. Thus we can write the following relation due to Lemma 1

$$\mu(\mathsf{F}_\beta \mathsf{A}_\alpha)^t(\ominus) \leq \frac{\mu(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\ominus)}{1 - \mu(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\odot)}, \qquad \text{for each } t \in \mathbb{N}.$$

Since $\mu \in \mathcal{A}_\ominus$ we can write μ as a convex combination of δ_{x^i} where $x^i \in \Delta$. And using the fact that E_α is linear we have

$$\begin{aligned} \tau_\mu &\leq \inf\{t \geq 0 : \mu(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\ominus) = 0\} \\ &= \inf\{t \geq 0 : \delta_{x^1}(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\ominus) = \delta_{x^2}(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\ominus) = \cdots = 0\} \\ &= \inf\{t \geq 0 : \delta_\xi(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\ominus) = 0\} \end{aligned} \tag{15}$$

where $\xi = x^{\mathsf{M}(\mu)}$.

Let us denote (15) by τ_ξ.

We know that $\delta_\xi(\mathsf{F}_\beta \mathsf{E}_\alpha)^t(\ominus)$ has a relation with the Markov Chain $(Z_t)_{t \in \mathbb{Z}_+}$. The process Z is the composition of the Markov Chains X and Y. Let us consider $Z_0 = \mathsf{M}(\mu)$.

We denote the absorption time of X and Y by

$$\tau^X_{\mathsf{M}(\mu)} = \inf\{t \geq 0 : X_t = 0 \text{ and } X_0 = \mathsf{M}(\mu)\} \text{ and } \tau^Y_{\mathsf{M}(\mu)} = \inf\{t \geq 0 : Y_t = 0 \text{ and } Y_0 = \mathsf{M}(\mu)\}$$

Since Z is the composition of X and Y, acting in this order, first X and then Y, in each step time, it follows that $\mathbb{E}\left(\tau^Z_{\mathsf{M}(\mu)}\right) \leq \min\left\{\mathbb{E}\left(\tau^X_{\mathsf{M}(\mu)}\right), \mathbb{E}\left(\tau^Y_{\mathsf{M}(\mu)}\right)\right\}$. Thus, from (10)

$$\mathbb{E}(\inf\{t \geq 0 : \delta_\xi \mathsf{F}_\beta^t = \delta_\oplus\}) = \mathbb{E}\left(\tau^X_{\mathsf{M}(\mu)}\right) \leq \sum_{i=1}^{\mathsf{M}(\mu)} \frac{1}{1-(1-\beta)^i} = g_\mu(\beta),$$

and

$$\mathbb{E}(\inf\{t \geq 0 : \delta_\xi \mathsf{E}_\alpha^t = \delta_\oplus\}) = \mathbb{E}\left(\tau^Y_{\mathsf{M}(\mu)}\right) = \frac{\mathsf{M}(\mu)}{\alpha} = f_\mu(\alpha).$$

Note that, if $\alpha \geq \mathsf{M}(\mu)/g_\mu(\beta)$, then $\min\{g_\mu(\beta), f_\mu(\alpha)\} = f_\mu(\alpha)$. On the other hand, if $\alpha < \mathsf{M}(\mu)/g_\mu(\beta)$, then $\min\{g_\mu(\beta), f_\mu(\alpha)\} = g_\mu(\beta)$.

Therefore,

$$\mathbb{E}(\tau_\mu) \leq \mathbb{E}(\tau_\xi) = \mathbb{E}\left(\tau^Z_{\mathsf{M}(\mu)}\right).$$

□

Proof of Theorem 2. Let us consider the initial distribution, δ_x, with $x \in \Delta$. In this case,

$$\mathsf{M}(\mu, t) = \mathsf{Pop}(x^t).$$

From the association among $(Z_t)_{t \in \mathbb{Z}_+}$, $\mathsf{F}_\beta \mathsf{E}_\alpha$ and $\mathsf{F}_\beta \mathsf{A}_\alpha$, it follows

$$\mathbb{P}\left(\mathsf{Pop}(x^t) \leq Z_t\right) = 1.$$

So,

$$\mathbb{E}(\mathsf{M}(\mu,t)) = \mathbb{E}(\mathsf{Pop}(x^t)) \le \mathbb{E}(Z_t).$$

Using identities $Z_0 = \mathsf{M}(\mu) = \mathsf{Pop}(x)$ and (14),

$$\mathbb{E}(\mathsf{M}(\mu,t)) \le (1-\beta)^{t-1}\left(\mathsf{M}(\mu) + \alpha\beta^{\mathsf{M}(\mu)} - \alpha\right).$$

□

5 Proof of Theorems 3 and 4

Theorem 3 collaborates with conditions of ergodicity of our operator, which are differents from that described in **(r3)**. Theorems 3 and **(r3)** give us the following relations:

- If $\alpha \le \beta$, then **(r3)** implies Theorem 3.
- If $\beta < \alpha$, then Theorem 3 implies **(r3)**.

Proof of Theorem 3. We get,

$$\mu\mathsf{F}_\beta\mathsf{A}_\alpha(\ominus) = \frac{\mu\mathsf{F}_\beta(\ominus) - \alpha\mu\mathsf{F}_\beta(\oplus,\ominus)}{1-2\alpha\mu\mathsf{F}_\beta(\oplus,\ominus)}.$$

From consistency of the measure, $\mu(\ominus) = \mu(\ominus,\ominus) + \mu(\oplus,\ominus)$. Thus, we conclude that $\mu(\ominus) \ge \mu(\oplus,\ominus)$. Consequently for all $\beta \in (0,1)$,

$$\mu\mathsf{F}_\beta(\oplus,\ominus) \le \mu\mathsf{F}_\beta(\ominus) \quad \text{and} \quad 1-2\alpha\mu\mathsf{F}_\beta(\oplus,\ominus) \ge 1-2\alpha\mu\mathsf{F}_\beta(\ominus).$$

Therefore

$$\begin{aligned}\frac{\mu\mathsf{F}_\beta(\ominus) - \alpha\mu\mathsf{F}_\beta(\oplus,\ominus)}{1-2\alpha\mu\mathsf{F}_\beta(\oplus,\ominus)} &\le \frac{\mu\mathsf{F}_\beta(\ominus) - \alpha\mu\mathsf{F}_\beta(\oplus,\ominus)}{1-2\alpha\mu\mathsf{F}_\beta(\ominus)} \\ &< \frac{\mu\mathsf{F}_\beta(\ominus)}{1-2\alpha\mu\mathsf{F}_\beta(\ominus)} \\ &= \frac{(1-\beta)\mu(\ominus)}{1-2\alpha(1-\beta)\mu(\ominus)}. \end{aligned} \tag{16}$$

Note that,

$$\begin{aligned}\frac{(1-\beta)\mu(\ominus)}{1-2\alpha(1-\beta)\mu(\ominus)} < \mu(\ominus) \quad &\Leftrightarrow \quad (1-\beta)\mu(\ominus) < \mu(\ominus) - 2\alpha(1-\beta)[\mu(\ominus)]^2 \\ &\Leftrightarrow \quad \mu(\ominus)[2\alpha(1-\beta)\mu(\ominus) - \beta] < 0. \end{aligned} \tag{17}$$

So (17) is true for $2\alpha(1-\beta)\mu(\ominus) - \beta < 0$. Thus, if $\mu(\ominus) \in (0, \beta/2\alpha(1-\beta))$, then (16) is less that $\mu(\ominus)$. So,

$$\mu\mathsf{F}_\beta\mathsf{A}_\alpha(\ominus) < \mu(\ominus).$$

Now, we are able to conclude that $\mu(\mathsf{F}_\beta\mathsf{A}_\alpha)^t(\ominus)$ is monotonously decreasing. Moreover this sequence has a lower bound, then $\mu(\mathsf{F}_\beta\mathsf{A}_\alpha)^t(\ominus) \to 0$, when $t \to \infty$. □

In [19] it was used two well known ideas: Peierls' contour method and duality planar graphs [4], which were used in the contact processes [5] and Stavskaya processes [18]. Here, the central idea is to count the probability to obtain a closed contour. At our contour exist four types of steps, which we denoted by 1,2,3 and 4. Each contour is composed by a quantity of horizontal steps,x, vertical steps, y; its last step is of the type k and has in total z steps. Now, we associate a rate to each step and denote by $S_k(x, y, z)$ the sum of rates of all contours with these x, y, z and k.

After implemented the methodology, the following inequality was attained, which will be our starting point.

$$\mu_t(\oplus) \leq \sum_{x=1}^{\infty}\sum_{y=0}^{\infty}\sum_{z=1}^{\infty} x \cdot [S_3(x, y, z) + S_4(x, y, z)]. \tag{18}$$

The numbers $S_k(x, y, z)$ satisfy the initial condition

$$S_k(x, y, 1) = \begin{cases} 1 & \text{if } x = 0, y = -1, \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and also satisfy the transition equations

$$\begin{cases} S_1(x, y, z+1) &=& S_1(x, y+1, z) + S_2(x, y+1, z) + S_3(x, y+1, z), \\ S_2(x, y, z+1) &=& 2\beta[S_1(x-1, y, z) + S_2(x-1, y, z) + S_3(x-1, y, z) + S_4(x-1, y, z)], \\ S_3(x, y, z+1) &=& \alpha \cdot [S_2(x+1, y-1, z) + S_3(x+1, y-1, z) + S_4(x+1, y-1, z)], \\ S_4(x, y, z+1) &=& (1-\alpha) \cdot [S_2(x, y-1, z) + S_3(x, y-1, z) + S_4(x, y-1, z)]. \end{cases} \tag{19}$$

To estimate (18), let us use sums

$$S_i(z) = \sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty} p^{-x} q^{-y} S_i(x, y, z) \qquad \text{with } i \in \{1, 2, 3, 4\}, \tag{20}$$

where p and q are positive parameters and p and q are such that $x < p^{-x}$ and $1 \leq q^{-y}$. Note that the right side of (18) has the following upper bound

$$\sum_{z=1}^{\infty}(S_3(z) + S_4(z)). \tag{21}$$

As in [11, 20], (20) satisty the initial conditions

$$S_1(1) = q, \qquad S_2(1) = S_3(1) = S_4(1) = 0,$$

From (19) and (20) the recurrence conditions are:

$$\begin{cases} S_1(z+1) &=& q(S_1(z) + S_2(z) + S_3(z)), \\ S_2(z+1) &=& \dfrac{2\beta}{p}(S_1(z) + S_2(z) + S_3(z) + S_4(z)), \\ S_3(z+1) &=& \dfrac{p\alpha}{q}(S_2(z) + S_3(z) + S_4(z)), \\ S_4(z+1) &=& \dfrac{1-\alpha}{q}(S_2(z) + S_3(z) + S_4(z)). \end{cases} \tag{22}$$

To write (22) in a matrix form, we introduce

$$S(z) = (S_1(z), S_2(z), S_3(z), S_4(z)).$$

Thus, the recurrence conditions are given by $S(z+1) = S(z)\cdot M$, i.e., $S(z) = S(1)\cdot M^{z-1}$ where M is the matrix

$$M = \begin{pmatrix} q & 2\beta/p & 0 & 0 \\ q & 2\beta/p & p\alpha/q & (1-\alpha)/q \\ q & 2\beta/p & p\alpha/q & (1-\alpha)/q \\ 0 & 2\beta/p & p\alpha/q & (1-\alpha)/q \end{pmatrix}.$$

We define

$$N = \begin{pmatrix} q & 2\beta/p & 0 & 0 \\ q & 2\beta/p & p\alpha/q & (1-\alpha)/q \\ q & 2\beta/p & p\alpha/q & (1-\alpha)/q \\ q & 2\beta/p & p\alpha/q & (1-\alpha)/q \end{pmatrix}.$$

As M and N are non negatives and $M \leq N$,

$$S(z) \leq S(1)\cdot N^{z-1}. \tag{23}$$

Note that N has four eigenvalues, where two among them are equal to zero. Moreover, its eigenvectors are generate by $(0,1,0,-1)$ and $(0,0,1,-1)$. The eigenvalue of Perron-Frobenius [15] is denoted by λ_{PF}. By end, we get λ_2.

$$\lambda_{PF} = \frac{q + 2\beta/p + (p\alpha + 1 - \alpha)/q + \sqrt{\Delta}}{2} \quad \text{and} \quad \lambda_2 = \frac{q + 2\beta/p + (p\alpha + 1 - \alpha)/q - \sqrt{\Delta}}{2},$$

where $\Delta = \left(q + \frac{2\beta}{p} + \frac{\alpha(p-1)+1}{q}\right)^2 - 4(\alpha(p-1)+1)$.

If $\lambda_{PF} < 1$, then $\beta < 11\alpha^2/200(3-\alpha)$. Moreover, fixed p, q and α, λ_{PF} is an increasing function in relation to β. Therefore if we assume

$$\beta < \frac{9\alpha^2}{1000}, \tag{24}$$

then $\lambda_{PF} < 1$. And this is the upper bound to β that we will adopt from now on.

The matrix N is diagonalizable, which allows us to write $N^z = Q^{-1}D^zQ$ (we are calculating eigenvectors from the left) where $Q = (q_{ij})_{i,j=1,2,3,4}$ is the eigenvectors matrix and $Q^{-1} = (\tilde{q}_{ij})_{i,j=1,2,3,4}$ is your inverse. We have $N^z = (n^z_{ij})_{i,j=1,2,3,4}$ where

$$n^z_{1j} = \lambda^z_{PF}\tilde{q}_{11}q_{1j} + \lambda^z_2\tilde{q}_{12}q_{2j}, \qquad \forall\ j = 1,2,3,4, \tag{25}$$

The eigenvectors associated with λ_{PF} and λ_2 are,

$$\left(\frac{\left(\lambda_i - \frac{\alpha(p-1)+1}{q}\right)q^2}{2\alpha\beta}v_3, \frac{\left(\lambda_i - \frac{\alpha(p-1)+1}{q}\right)q}{p\alpha}v_3, v_3, \frac{1-\alpha}{p\alpha}v_3\right) \quad \text{for } v_3 \in \mathbb{R}\backslash\{0\} \text{ and } i \in \{PF, 2\}. \tag{26}$$

We will take positive unitary representatives of the eigenvectors of λ_{PF} and λ_2 by the expression in (26). Due to our choise of eigenvectors, $\sum_{j=1}^{4} q_{1j} = \sum_{j=1}^{4} q_{2j} = 1$. We can verify

$$\tilde{q}_{11} = \frac{q_{22} + q_{23} + q_{24}}{\xi} \quad \text{and} \quad \tilde{q}_{12} = -\frac{q_{12} + q_{13} + q_{14}}{\xi},$$

where $\xi = q_{11}(q_{22}+q_{23}+q_{24}) - q_{21}(q_{12}+q_{13}+q_{14}) = q_{11}-q_{21}$.

Therefore,

$$\tilde{q}_{11} = \frac{1-q_{21}}{q_{11}-q_{21}} \qquad \text{and} \qquad \tilde{q}_{12} = \frac{q_{11}-1}{q_{11}-q_{21}}. \tag{27}$$

Thus, $\tilde{q}_{11}+\tilde{q}_{12}=1$.

The eigenvectors associated with λ_{PF} and λ_2 are unitaries. So,

$$v_3 = \frac{2\alpha\beta/q}{(\lambda_i - (\alpha(p-1)+1)/q))(q+2\beta/p) + 2\beta/p(\alpha(p-1)+1)/q)} \quad \text{for } i \in \{PF, 2\}.$$

Assuming (24) and using (23) and (25),

$$\begin{aligned}
\sum_{z=1}^{\infty}(S_3(z)+S_4(z)) &\leq q\left(\sum_{z=0}^{\infty}(n_{13}^z + n_{14}^z)\right) \\
&= q\left(\sum_{z=0}^{\infty}(\lambda_{PF}^z\tilde{q}_{11}q_{13} + \lambda_2^z\tilde{q}_{12}q_{23}) + \sum_{z=0}^{\infty}(\lambda_{PF}^z\tilde{q}_{11}q_{14} + \lambda_2^z\tilde{q}_{12}q_{24})\right) \\
&= q\left(\frac{\tilde{q}_{11}(q_{13}+q_{14})}{1-\lambda_{PF}} + \frac{\tilde{q}_{12}(q_{23}+q_{24})}{1-\lambda_2}\right) \\
&= q\left(\frac{\tilde{q}_{11}(q_{13}+q_{14})}{1-\lambda_{PF}} + \frac{(1-\tilde{q}_{11})(q_{23}+q_{24})}{1-\lambda_2}\right),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
q_{11} &= \frac{q\lambda_{PF} - (\alpha(p-1)+1)}{\lambda_{PF}(q+2\beta/p) - (\alpha(p-1)+1)}, \quad & q_{21} &= \frac{q\lambda_2 - (\alpha(p-1)+1)}{\lambda_2(q+2\beta/p) - (\alpha(p-1)+1)}, \\
q_{13} &= \frac{2\alpha\beta/q}{\lambda_{PF}(q+2\beta/p) - (\alpha(p-1)+1)}, \quad & q_{23} &= \frac{2\alpha\beta/q}{\lambda_2(q+2\beta/p) - (\alpha(p-1)+1)}, \\
q_{14} &= \frac{2\beta(1-\alpha)/pq}{\lambda_{PF}(q+2\beta/p) - (\alpha(p-1)+1)}, \quad & q_{24} &= \frac{2\beta(1-\alpha)/pq}{\lambda_2(q+2\beta/p) - (\alpha(p-1)+1)}.
\end{aligned}$$

We denote (28) by

$$LS(p, q, \beta, \alpha). \tag{29}$$

Proof of (A. 4). Let us consider $p = 3/10$, $q = 1-\alpha/3$ with $\alpha \in (0,1)$. Thus, (29) is an increasing function in relation to β, for $\beta \in [0, 9\alpha^2/1000]$. So, under these conditions, if $LS(3/10, 1-\alpha/3, 9\alpha^2/1000, \alpha)$ is less than 1, then $LS(3/10, 1-\alpha/3, \beta, \alpha)$ is less than 1.

If $\beta = 9\alpha^2/1000$ then (29) is only function of α, which through some algebraic computation is expressed by

$$\begin{aligned}
&LS(3/10, 1-\alpha/3, 9\alpha^2/1000, \alpha) \\
&\quad = \frac{-(36(-3+\alpha)\alpha^2(9\alpha^3 - 77\alpha^2 + 465\alpha - 450))}{(-9\alpha^3 + f(\alpha)\alpha + 77\alpha^2 - 3f(\alpha) - 315\alpha)(9\alpha^3 + f(\alpha)\alpha - 77\alpha^2 - 3f(\alpha) + 315\alpha)}.
\end{aligned}$$

This expression is continuous and monotonic decreasing for $\alpha \in [0,1]$. Moreover, it assumes value less than 1 for $\alpha = 0$ and greater than zero for $\alpha = 1$. □

Proof of (B. 4). Assuming $R = \{\oplus\} \subset \mathbb{A}^{\mathbb{Z}}$, i. e., $R = \{x \in \mathcal{A}^{\mathbb{Z}} : x_{i_0} = \oplus$ for some $i_0 \in \mathbb{Z}\}$. We denote $\delta = LS(p, q, \beta, \alpha)$ and let us consider $p = 3/10$, $q = 1 - \alpha/3$ and $\beta \in (0, 9\alpha^2/1000)$. Under these conditions and using the Theorem 4 item (A. 4),

$$\mu_t(\oplus) = \delta_{\ominus}(\mathsf{F}_\beta \mathsf{A}_\alpha)^t(\oplus) < LS(p, q, \beta, \alpha) \qquad \text{for all } t.$$

From **Toom et. al., (2011)** the operator $\mathsf{F}_\beta\mathsf{A}_\alpha$ has an invariant measure ν, such that

$$\nu(\oplus) < LS(p, q, \beta, \alpha) < 1.$$

□

For $\beta < 9\alpha^2/1000$ the operator $\mathsf{F}_\beta\mathsf{A}_\alpha$ has at least two distinct invariant measures, namely the measure $\delta_\oplus$ and the measure ν of the Theorem 4 item (B. 4).

6 Computational study

Computational simulations are widely used in the study of particles systems interacting[12, 1, 3, 17, 16, 13] and allows us to theorize what occurs in certain situations. Our process may be defined on a finite space with periodic boundary conditions, $\mathbb{Z}_n$ the set of remainders modulo n, where n is an arbitrary natural number. Let us consider the set of states $\Omega_n = \{\ominus, \oplus\}^{\mathbb{Z}_n}$. We call elements of Ω_n periodic configurations. The periodic configurations are finite sequences of minuses and pluses, now we imagine these sequences to have a periodic form.

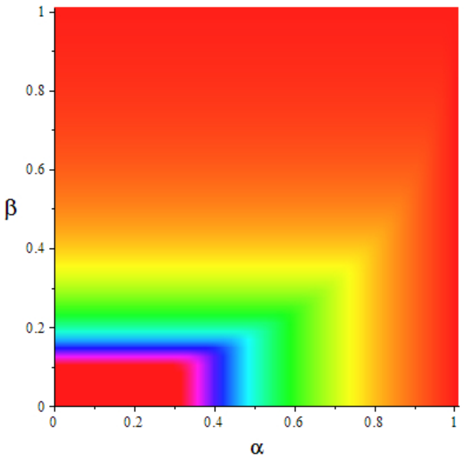
Considering $\mu \in \mathcal{A}_\ominus$ to estimate $\mathbb{E}(\tau_\mu)$, we will perform a computational study. Let us consider only δ-measures, whose $x \in \Delta$, i.e., x has a finite number of minuses in an ocean of pluses and $\mathsf{M}(\delta_x) \in \{10, 20, \ldots, 100\}$. In this case, $\mathsf{M}(\delta_x)$ indicates the number of minuses on the configuration x. For the computational study, initially we take $n = 1000$. We fix the parameters $(\alpha, \beta) \in \{0.01, 0.02, \ldots, 1\}^2$ and perform our Monte Carlo study for 100000 time steps or when all the components attain state plus. We call this exact moment, *stop time*.

We performed our computational study within a cycle with a fixed pair (α, β): After repeat the simulation 100 times, we will save 100 stop times which we denote by $t^1, t^2, t^3, \ldots, t^{100}$, the t^i denotes the stop time on the i-th repetition. Using these information we are able to evaluate

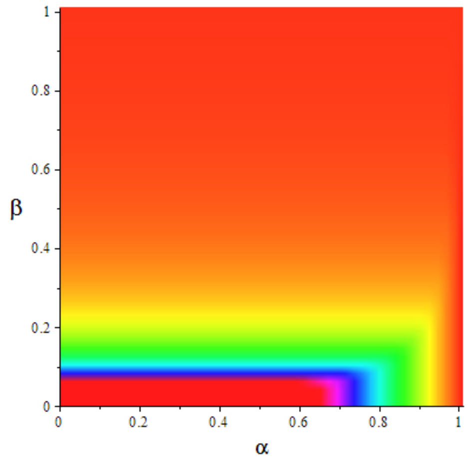
$$\widehat{\mathbb{E}}(\tau_\mu) = \frac{\sum_{i=1}^{100} t^i}{100},$$

our estimator of $\mathbb{E}(\tau_\mu)$.

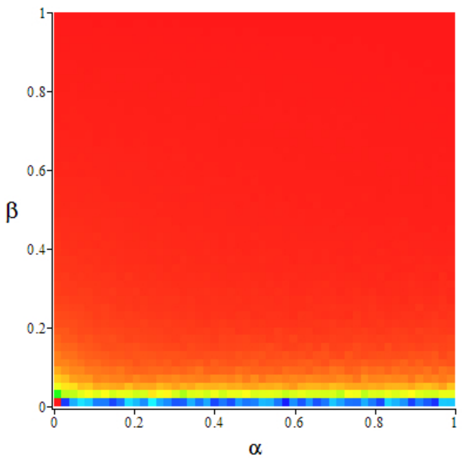
In Figure 2 we stated $\mu = \delta_x$ whose $\mathsf{Pop}(x) \in \{10, 100\}$. We present the values of the upper bound of $\mathbb{E}(\tau_\mu)$ obtained from computer simulations and its upper bound from the Theorem 1. The upper bound values of $\mathbb{E}(\tau_\mu)$, are represented by colors. In all the figures, 2(a) - 2(d) for a fixed α and making β grow, the upper bound value decrease. In 2(a) and 2(b) we use Theorem 1 and in the Figures 2(c) and 2(d) we show our results from computer studies. In each case, we describe the minimum and maximum mean time observed. As expected, the minimum value is obtained on the region where α and β are close to zero(dark red region) and the maximum value is obtained on the region where β is close to one(light red region). Between these regions, which exhibits small and big value of mean time of convergence, we get a intermediary ones represented by colors not



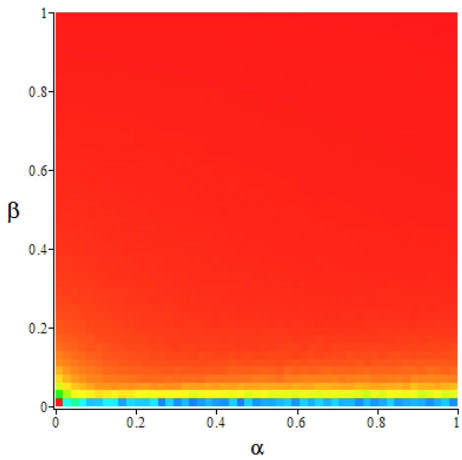
(a) Upper bounds of $\mathbb{E}(\tau_\mu)$ with $\mathsf{Pop}(x) = 10$, the minimum mean time equal to 10.0000 and the maximum mean time equal to 296.4759.



(b) Upper bounds of $\mathbb{E}(\tau_\mu)$ with $\mathsf{Pop}(x) = 100$, the minimum mean time equal to 100.0000 and the maximum mean time equal to 570.3338.



(c) The computation estimation of $\mathbb{E}(\tau_\mu)$ with $\mathsf{Pop}(x) = 10$, the minimum mean time equal to 1.0101 and the maximum mean time equal to 285.6263.



(d) The computation estimation of $\mathbb{E}(\tau_\mu)$ with $\mathsf{Pop}(x) = 100$, the minimum mean time equal to 1.0101 and the maximum mean time equal to 519.0707.

Figure 2: Graphics from numerical studies of $\mathbb{E}(\tau_\mu)$.

red. This intermediary region is really sharp, when we see the results obtained through computer simulations.

Figure 3 shows the rigorous estimation in [19, 11, 20] ($\beta < \alpha^2/250$); the obtained by us on the Theorem 4, ($\beta < 9\alpha^2/1000$) and the "curve" (well, it is not exactly a curve, it is somewhat fuzzy) corresponding for the pars (α, β) from the computer simulations. One can see with more details in [12] the procedure performed, by one of us, to simulate this process. The procedure required some specific adjustment once that on average the finite process, which we simulate, " disappear ". Heuristically speaking, the distance between the numerical curve and the theoretical ones was expected. Since that the methodology of contour applied to obtain the theoretical curve takes an upper bound for the number contour.

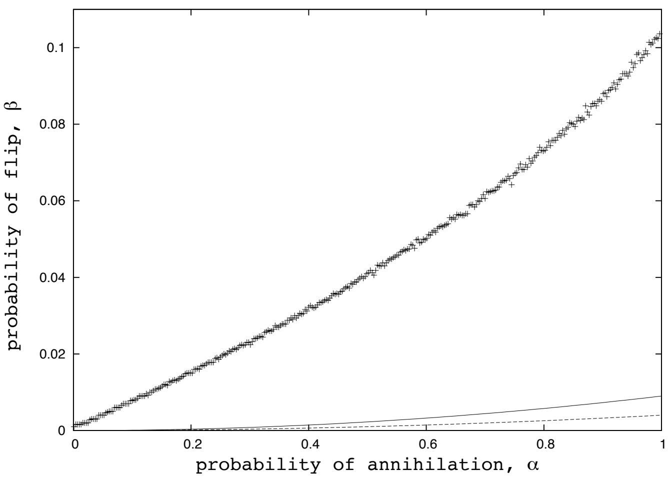


Figure 3: Here, we describe the curves in which $\beta = \alpha^2/250$ and $\beta = 9\alpha^2/1000$. Also, we plot the datas obtained by computational simulation of the Flip-Annihilation process. These data estimate the critical curve.

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