Locality of the interaction affects dynamics in probabilistic cellular automata

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January 11, 2022

Abstract

We study a class of one-dimensional probabilistic cellular automata, in which each component can be in either state *zero* or *one*. The component interacts with two neighbors, if its neighbors are in an equal state, then the component assumes the same state as its neighbors. If its neighbors are in different states, the following can happen: a *one* on the right side of a *zero*, in which case the component becomes *one* with probability α or *zero* with probability $1-\alpha$; conversely, a *zero* on the right side of an *one* in which case, the component becomes *one* with probability β or *zero* with probability $1-\beta$. For a set of initial distributions when both neighbors are placed on the right- side (respectively both on the left- side) of a component, we prove that the process always converges weakly to the measure concentrated on the configuration where all the components are zeros. When one neighbor is placed on the left side and the other on the right side, the same convergence happens when $\beta < f_N(\alpha)$, where N is the distance between the neighbors. However, this convergence does not happen for $\beta > 1/2\alpha$. Thus, in this case, we get the regimes of ergodicity and non-ergodicity. Moreover, we exhibit another type of phase transition, independent of neighbors' locations. We also, present some numerical studies, in which we use mean field approximation and Monte Carlo simulation.

Key words: Probabilistic cellular automata, ergodicity, Phase transition.

1 Statements

A class of one-dimensional probabilistic cellular automata, PCA by simplicity, is considered. Cases in which a random process with local interaction exhibits some form of non-ergodicity remain non-trivial and , as a result, continue to attract attention. As a result, ergodicity [8] and the invariant measure [1] are of particular interest. Some research has been conducted on time required to reach the equilibrium,i.e., the invariant probability measure, for PCA in finite or infinite space [4, 6, 7, 10]. Generally, the studies about PCA consider its interaction with the nearest neighbors. However, this assumption is not the only possible one. Here we present

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another approach. In part, that assumption is motivated by recent models in more recently formulated theories in sciences like biology, economy and social science [2, 3, 5], establishing the need for a non-nearest interaction among its components.

The study of PCA is linked with several research areas: statistical physics and theoretical computer science, each with a different viewpoint [13]. This study presents various algorithmically unsolvable problems[14]; among them, we give attention to the undecidability of the problem of ergodicity, i. e., there exists no algorithm, when taking in input the parameters of a PCA if it is ergodic or not. When we face an undecidable problem, we work closely to the boundaries of natural possibility. This encourages us to treat the partial results we have obtained with more respect. A PCA is ergodic if its action on probability measures has a unique invariant measure and for any initial distribution the process converges weakly for it. From the initial studies of the Ising model, it became a tradition among physicists to discern a qualitative difference between one-dimensional and multi-dimensional processes for all models with local interaction. This is stated in the shape of the "positive rates conjecture", which was refuted by Peter Gács [15]. However, it remains nontrivial when a random process with one-dimensional local interaction shows some form of non-ergodicity, and for this reason, it attracts attention [16].

We studied a PCA that is not restricted to the nearest-neighbor interaction, in which each component can have two states (0 (zero) or 1 (one)), and they interact with two neighbors (not necessarily the closest ones). Informally, when its neighbors are in the same state (both zero or one), the component assumes the neighbour's state. If its neighbors are in the states 0 (zero) and 1 (one), the component will either become 1 (one) with probability α or 0 (zero) with probability $1-\alpha$. Now, its neighbors assume 1 (one) and 0 (zero), respectively, and the component becomes 1 (one) with probability β or 0 with probability $1-\beta$.

Given a set of possible initial measures for the process, we showed that the process converges to the measure concentrated on the configuration where all the components are zero: (i) when both of its neighbors are located on the left- side(respectively in its right- side) and (ii) when β is less than equal to a function of α and N. Also, there are two regions on the parameter space of the process: one where the mean time of convergence is always infinite, and the other where it is finite. Finally, we used mean-field approximation to study the process.

The random operators of interest are defined on the configuration space $\Omega = \{0, 1\}^{\mathbb{Z}}$, where $\mathbb{Z}$ is the set of integer numbers, and 0 and 1 are called *one* and *zero*, respectively. Each $x \in \Omega$, called configuration, is a bi-infinite sequence of *zeros* and *ones*. A configuration $x \in \Omega$ is determined by its components x_i for each $i \in \mathbb{Z}$ and $x_i \in \{0, 1\}$. The configuration x, whose components are *zeros* or *ones*, is called *all zeros* and *all ones*. Two configurations x and y are close to each other if the set $\{i \in \mathbb{Z} \quad : x_i \neq y_i\}$ is finite. A configuration is called the island of ones if it is close to *all zeros*, and we denote the set of islands of ones by Δ. If $x \in \Delta$ and x is not all zeros, then there are positions $i < j$ such that $x_{i+1} = x_{j-1} = 1$ and $x_k = 0$ if $k \leq i$ or $j \leq k$. For this case, we define the length of the island x by $j - i - 1$, and we denote this by $\mathsf{length}(x)$.

The normalized measures concentrated in the configurations *all zeros* and *all ones* are denoted by δ_0 and δ_1, respectively. Also, given a configuration x, we denote the normalized measure concentrated in x by δ_x. We define a cylinder in Ω in the usual way and we denote any *thin cylinder* set

$$\{x \in \Omega : x_i = a_i, \text{ for all } i \in I\},$$

where $(a_i)_{i \in I} \in \{0, 1\}$ and I is finite subset of $\mathbb{Z}$. By simplicity, we denote *thin cylinder* by $\{x_i = a_i, i \in I\}$. We denote by $\mathcal{M}$ the set of normalized measures on the σ-algebra generated by cylinders in Ω. By convergence in $\mathcal{M}$ we mean convergence on all cylinders. We say that $\mu \in \mathcal{M}$ is *uniform* if it is invariant under space shifts, i.e., $\mu(x_i = a) = \mu(x_{i+j} = a)$ for any

$j \in \mathbb{Z}$. We denote by $\mathcal{A}$ the set of normalized measures in Δ. A measure belonging to $\mathcal{A}$ is called an archipelago of ones. As Δ is countable any $\mu \in \mathcal{A}$ is a convex combination of δ_x-measures where $x \in \Delta$. So, $\mathcal{A} \subset \mathcal{M}$.

Given an operator P, $\mathsf{P} : \mathcal{M} \to \mathcal{M}$, and an initial measure $\mu \in \mathcal{M}$, the sequence $\mu, \mu\mathsf{P}, \mu\mathsf{P}^2, \ldots$ define the random process. We say that a probability measure μ is *invariant* under P if $\mu\mathsf{P} = \mu$. For each $i \in \mathbb{Z}$, we call $V(i) \subset \mathbb{Z}$ the set of neighbors of i. We define the translation operator $\mathsf{T}_p : \Omega \to \Omega$ as follows

$$(x\mathsf{T}_p)_k = x_{k-p}, \text{ for all } k \in \mathbb{Z},$$

which induces translation at measures. Throughout the text, we use the same notation, T_p to indicate translation in both cases: measures and configurations.

We will consider a class of probabilistic cellular automata in $\mathbb{Z}$, where for each $k \in \mathbb{Z}$, $V(k) = \{k + p, k + q\}$. It is well-known[9, 12, 11] that the operator P is determined by the transition probabilities $\theta(b_k|a_{V(k)}) \in [0, 1]$, such that

$$\theta(b_k|a_{k+p}a_{k+q}) \in [0, 1] : \sum_{b_k \in \{0,1\}} \theta(b_k|a_{k+p}a_{k+q}) = 1.$$

Let $x, y \in \Omega$. A general operator P with $V(k) = \{k + p, k + q\}$ is defined

$$\mu\mathsf{P}(y_i = b_i, i \in I) = \sum_{a_j,\ j\ \in\ V(I)} \mu(x_i = a_i, i \in V(I)) \prod_{i \in I} \theta(b_i|a_{i+p}a_{i+q}), \tag{1}$$

where $V(I) = \bigcup\limits_{i \in I} V(i)$. For fixed values of p and q we denote $\mathsf{F}_{(p,q)}$ the operator whose transition probabilities are

$$\theta(1|a_{k+l}a_{k+r}) = \begin{cases} a_{k+p} & \text{if} \quad a_{k+p} = a_{k+q}; \\ \alpha & \text{if} \quad a_{k+l} = 0 \text{ and } a_{k+r} = 1; \\ \beta & \text{if} \quad a_{k+l} = 1 \text{ and } a_{k+r} = 0. \end{cases} \tag{2}$$

Where $l = \min\{p, q\}$, $r = \max\{p, q\}$ and α, β belongs to $[0, 1]$. We do not consider $N = 0$, i.e., $p = q$, once it is a simple case. For technical reasons: If p and q are non-positive, then we assume that $q < p$; and if p or q is positive, then $p < q$. $\mathsf{F}_{(p,q)}$ denotes by F when $p = 0$ and $q = 1$. Figure 1 depicts how the interaction occurs.

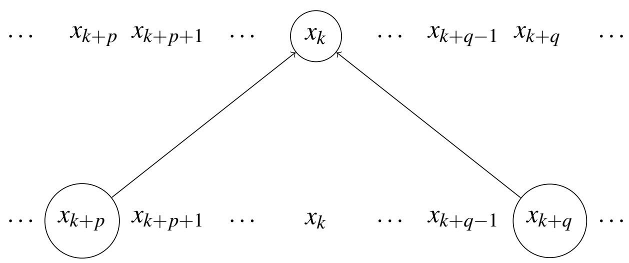

Figure 1: *The non-nearest interaction of the process considering $p < q$.*

From now onwards throughout the text the measures μ will denote $\mu \in \mathcal{A}$. At the text $\mu(1)$ means the density of ones at μ. Thus, $\mu(1) = 0$ for $\mu = \delta_0$ and $\mu(1) > 0$ for $\mu \neq \delta_0$. The measures δ_0 and δ_1 are invariant under $\mathsf{F}_{(p,q)}$. Given p, q and μ we define the random variable

$$\tau_{\mu}^{(p,q)} = \inf\{t \geq 0 : \mu\mathsf{F}_{(p,q)}^t(1) = 0\}. \tag{3}$$

The infimum of the empty set is ∞. When $p = 0$ and $q = 1$, we denote $\tau_\mu^{(p,q)}$ by τ_μ. The random variable in $\tau_\mu^{(p,q)}$ indicates the time to attain δ_0 having $\mathsf{F}_{(p,q)}$ started to measure μ. Given μ there are islands of ones x^i for $i \in \mathbb{N}$ such that $\mu = \sum_{i \in \mathbb{N}} k_i \delta_{x^i}$ where $k_1 > 0, k_2 > 0, \ldots$ and $k_1 + k_2 + \ldots = 1$. We denote the expected value by $\mathbb{E}$.

Now we describe some trivial cases, in which we verify immediately: $\alpha \in \{0,1\}$ or $\beta \in \{0,1\}$. If $\alpha = \beta = 0$, $\alpha = 0$ and $\beta \in (0,1)$ or $\alpha \in (0,1)$ and $\beta = 0$, then the process converge to the measure δ_0. If $\alpha = \beta = 1$, $\alpha = 1$ and $\beta \in [0,1)$ or $\alpha \in [0,1)$ and $\beta = 1$, resulting in the density of ones at each time step is always positive. Therefore, these extreme cases will not be considered.

Let us denote

$$f_N(\alpha) = (1-\alpha)^N - N + \frac{1-(1-\alpha)^N}{\alpha},$$

where N is the euclidian distance: precise $|p - q|$.

Theorem 1. *Given $\mu \in \mathcal{A}$, $\quad 0 < \alpha < 1$ and $0 < \beta < 1$. If p, q are non-negative values (respectively non-positive values), then*

$$\lim_{t \to \infty} \mu \boldsymbol{F}^t_{(p,q)} = \delta_0.$$

Theorem 2. *Given $\mu \in \mathcal{A}$, $\quad 0 < \alpha < 1$ and $\beta < f_N(\alpha)$. If $p < 0 < q$, then*

$$\lim_{t \to \infty} \mu \boldsymbol{F}^t_{(p,q)} = \delta_0.$$

Theorem 3. *Given $\mu \in \mathcal{A}$, $\quad 0 < \alpha < 1$ and $\beta > 1/2\alpha$. If $p < 0 < q$, then*

$$\lim_{t \to \infty} \mu \boldsymbol{F}^t_{(p,q)} \neq \delta_0.$$

Theorem 1 shows the ergodicity of the process when the neighbors are on the same side (left or right). In contrast, theorems 2 and 3 represent a phase transition between ergodicity and non-ergodicity. Hence, the locality of the interaction can drastically change the dynamic of the process.

Theorem 4. *Given $\mu \in \mathcal{A}$, $\quad \alpha, \beta \in (0,1)$.*
$(A.4)$ *If $\beta \geq f_1(\alpha)$, then $\mathbb{E}(\tau_\mu^{(p,q)}) = \infty$;*
$(B.4)$ *If $\beta < f_N(\alpha)$, then $\mathbb{E}(\tau_\mu^{(p,q)}) < \infty$.*

In order to prove the item $(B.4)$, we need to use a significant quantity of technical tools; we chose to put only its prove in section 4.

Theorem 4 gives us a kind of phase transition, independent of the locality of neighbors.

2 Order

Let us consider $0 < 1$. We shall introduce a partial order on $\{0,1\}^{\mathbb{Z}}$. Given two configurations x and y, we say that x *precedes* y, or what is the same, y *succeeds* x respectively when $x \prec y$ or $y \succ x$ for all $i \in \mathbb{Z}$.

We consider that a measurable set $S \subset \{0,1\}^{\mathbb{Z}}$ is *up-set* if

$$\forall x, y \in \Omega, (x \in S \text{ and } x \prec y) \implies y \in S.$$

Notably, in every up-set S, the configuration *all ones* belong to S. So, the set $S = \{1\}^{\mathbb{Z}}$ is up-set. Also, the set $S = \Omega^{\mathbb{Z}}$ is up-set.

We introduced a partial order on $\mathcal{M}$ such that μ precedes ν as we denote $\mu \prec \nu$ (respectively, ν succeeds μ we denote $\nu \succ \mu$), if $\mu(S) \leq \nu(S)$ for any up-set S (or $\mu(S) \geq \nu(S)$ for all lower S, complementary set of a up-set).

We call an operator $\mathsf{P} : \mathcal{M} \to \mathcal{M}$ monotone if $\mu \prec \nu$ implies $\mu\mathsf{P} \prec \nu\mathsf{P}$. Lemma 1 can be found in [11, 12, 4].

Lemma 1. *For all configurations x and y, an operator P on $\{0,1\}^{\mathbb{Z}}$ with transition of probabilities $\theta(\cdot|\cdot)$ is monotone if and only if*

$$x \prec y \implies \theta(1|x_{k+p}x_{k+q}) \leq \theta(1|y_{k+p}y_{k+q}) \tag{4}$$

Lemma 2. *Given p and q, the operator $\mathsf{F}_{(p,q)}$ is monotone.*

Proof. Using Lemma 1 and the definition of $\mathsf{F}_{(p,q)}$ in (2).

$\blacksquare$

3 Proof of Theorems 1, 2, and 4 items (A.4)

We consider that a configuration x is a *jump configuration* if there is a position i for which $x_j = 1$ for all $j < i$ and $x_j = 0$ for all $j \geq i$ by simplicity, we say the configuration is $(10, i)$-*jump* and $\mathcal{J}_{10}^{i}$ denotes jump and its concentrated measure. Analogously, we say that a configuration x is $(01, i)$-*jump* if there is position i such that $x_j = 0$ for all $j < i$ and $x_j = 1$ for all $j \geq i$. We denote the measure concentrated in $(01, i)$-*jump* by $\mathcal{J}_{01}^{i}$. Figure 2 illustrates $(10, -1)$-jump and $(01, -1)$-jump.

$$\begin{array}{ccccccccl}
\cdots & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \quad (10,-1)\text{-}jump \\
\cdots & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \cdots \quad x \\
\cdots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \quad (01,-1)\text{-}jump \\
\cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots \\
 & x_{-3} & x_{-2} & x_{-1} & x_0 & x_1 & x_2 & x_3 &
\end{array}$$

Figure 2: *We illustrate a $(10, -1)$-jump, a $(01, -1)$-jump, and a configuration x, which succeeds $(01, -1)$-jump.*

The pseudo-code, algorithm 1, shows the coupling between two stochastic processes generated by the operators F and $\mathsf{F}_{(p,q)}$, with $p,\ q$ non-negative (analogously non-positive) and having the same initial condition. The variables $x(j,t)$ and $y(j,t)$ are components of two marginal densities at both position j and time t, this type of coupling is extensively used in [9].

Algorithm 1: COUPLING BETWEEN THE PROCESSES GENERATED BY F AND $\mathsf{F}_{(p,q)}$ WITH $q-p>0$

1 **Do** *cont* $\leftarrow 0$ and $q-p>0$
2 **fixed** $i \in \mathbb{Z}$
3 **for** *each* $j \in \mathbb{Z}$ **do**
4 $\quad x(j,0) \leftarrow \begin{cases} 1 & \text{if } j \leq i \\ 0 & \text{if } j > i \end{cases}$
5 $\quad y(j,0) \leftarrow x(j,0)$
6 **end**
7 **for** $t \in \mathbb{N}$ **do**
8 $\quad$ **for** $j \in \mathbb{Z}$ **do**
9 $\quad\quad U_j^t \sim \mathcal{U}_{[0,1]}$
10 $\quad$ **end**
11 $\quad$ **for** $j \in \mathbb{Z}$ **do**
12 $\quad\quad x(j,t) \leftarrow \begin{cases} x(j,t-1) & \text{if } j \neq i \\ 0 & \text{if } j = i \text{ and } U_j^t > \beta \\ x(j,t-1) & \text{if } j = i \text{ and } U_j^t \leq \beta \end{cases}$
13 $\quad\quad y(j,t) \leftarrow \begin{cases} y(j+p,t-1) & \text{if } y(j+p,t-1) = y(j+q,t-1) \\ 1 & \text{if } y(j+p,t-1) < y(j+q,t-1) \text{ and } U_j^t < \alpha \\ 0 & \text{if } y(j+p,t-1) < y(j+q,t-1) \text{ and } U_j^t > \alpha \\ 1 & \text{if } y(j+p,t-1) > y(j+q,t-1) \text{ and } U_j^t < \beta \\ 0 & \text{if } y(j+p,t-1) > y(j+q,t-1) \text{ and } U_j^t > \beta \end{cases}$
14 $\quad$ **end**
15 $\quad$ **if** $x(i,t) = y(i,t) = 0$ **then** *cont* $\leftarrow$ *cont* $+ 1$ and $i \rightarrow i-1$
16 **end**

Lemma 3. *For $i \in \mathbb{Z}$ and $p,\ q$ non-negative,*

$$\mathcal{J}_{10}^{i}\mathsf{F}_{(p,q)}^{t}(1) \leq \mathcal{J}_{10}^{i}\mathsf{F}^{t}(1).$$

Proof. In the pseudo-code, lines $4-5$, we describe the initial condition $\mathcal{J}_{10}^{i}$. Lines $12-13$ describe the action of the operators F and $\mathsf{F}_{(p,q)}$, respectively. Clearly, we see that $\mathbb{P}(y(j,t) \leq x(j,t)) = 1$ for each $t \in \mathbb{N}$.

■

Lemma 4. *If $\beta \in (0,1)$, then*

$$\lim_{t\to\infty} \mathcal{J}_{10}^{i}\mathsf{F}^{t}(1) = 0.$$

Proof. At the pseudo-code, line 15, *cont* is the random variable where your value at the t-th step is given by $\sum_{k=1}^{t} L_k^{1-\beta}$, where $L_1^{1-\beta}, L_2^{1-\beta}, \ldots$ is a sequence of independent and identically distributed random variables, whose $\mathbb{P}(L_1^{1-\beta} = 1) = 1-\beta$ and $\mathbb{P}(L_1^{1-\beta} = 0) = \beta$. So, using the strong law of Kolmogorov $\sum_{k=1}^{t} L_k^{1-\beta}/t$ converges almost surely to $1-\beta$ when t tends to infinity.

■

Proof of Theorem 1. First, we shall prove the theorem for $\delta-$measures. We consider $q-p>0$. The case $q-p<0$ is analogous. From Lemma 3

$$\mathcal{J}^i_{10}\mathsf{F}^t_{(p,q)}(1) \leq \mathcal{J}^i_{10}\mathsf{F}^t(1).$$

From Lemma 4, for a given $i \in \mathbb{Z}$,

$$\lim_{t\to\infty} \mathcal{J}^i_{10}\mathsf{F}^t(1) = 0 \implies \lim_{t\to\infty} \mathcal{J}^i_{10}\mathsf{F}^t_{(p,q)}(1) = 0.$$

Note that given δ_x, there is $i \in \mathbb{Z}$ such that $\delta_x \prec \mathcal{J}^i_{10}$. So, by monotonicity of operator $\mathsf{F}_{(p,q)}$,

$$\lim_{t\to\infty} \delta_x\mathsf{F}^t_{(p,q)}(1) = 0.$$

Now, we prove for any archipelago μ. The measure μ can be written as a convex combination of $\delta_{x^1}, \delta_{x^2}, \ldots$, where $x^1, x^2 \ldots$ belongs to Δ. As proved previously, for a fixed $i \in \mathbb{N}$

$$\lim_{t\to\infty} \delta_{x^i}\mathsf{F}^t_{(p,q)} = \delta_0.$$

The $\mathsf{F}_{(p,q)}$ is linear, so $\mu\mathsf{F}_{(p,q)}$ is a convex combination of $\delta_{x^1}\mathsf{F}_{(p,q)}, \delta_{x^2}\mathsf{F}_{(p,q)}, \ldots$. Thus, we get that $\lim_{t\to\infty} \mu\mathsf{F}^t_{(p,q)} = \delta_0$.

■

Note that

$$\lim_{t\to\infty} \mathcal{J}_{01}\mathsf{F}^t_{(p,q)} = \begin{cases} \delta_0, & \text{if } p \text{ and } q \text{ are both non-positive;} \\ \delta_1, & \text{if } p \text{ and } q \text{ are both positive.} \end{cases}$$

It shows that neighbors' position on the interaction effects on system behavior.

Proof of Theorem 2. It is a direct consequence of item (B.4) of Theorem 4.

If $\mathsf{length}(x) = 1$, then $\delta_x\mathsf{F}^t_{(p,q)}(1)$ is the distribution of X_t, where $(X_t)_{t\in\mathbb{N}}$ is a birth and death process whose $X_0 = 1$.

More specifically,

$$\begin{array}{rcl} \mathbb{P}(X_t = a+1|X_{t-1} = a) & = & \theta(1|01)\theta(1|10); \\ \mathbb{P}(X_t = a-1|X_{t-1} = a) & = & \theta(0|01)\theta(0|10); \\ \mathbb{P}(X_t = a|X_{t-1} = a) & = & \theta(1|01)\theta(0|10) + \theta(0|01)\theta(1|10). \end{array} \tag{5}$$

Where $a \in \mathbb{N}$.

We denote by h_i, the absorption probability of the process X hits the state zero considering that we started at state i and the hitting time of state zero given that we started at state i by

$$H_i = inf\{t \geq 0 : X_t = 0 \text{ and } X_0 = i\}.$$

At our case, if $\beta < f_1(\alpha)$, then $h_i = 1$ and if $\beta \geq f_1(\alpha)$, then $h_i < 1$. Thus $\mathbb{E}(H_i) = \infty$, for all $i \geq 1$

Lemma 5. *Given $x \in \Delta$ such that $x = \underline{x}$. If $\beta > f_1(\alpha)$, then $\mathbb{E}(\tau_x^{(p,q)}) = \infty$.*

Proof. Let us consider the birth and death process, $(X_t)_{t\in\mathbb{N}}$ where $X_0 = 1$ and X_t has distribution $\delta_x \mathsf{F}^t_{(p,q)}$ considering $\mathsf{length}(x) = 1$. So,

$$\mathbb{E}\left(\tau_x^{(p,q)}\right) = \mathbb{E}(H_1^X).$$

Using the previous observations we get $\mathbb{E}\left(\tau_x^{(p,q)}\right) = \infty$.

■

For any $\mu = \sum_{i=1}^{\infty} k_i \delta_{x^i}$, we define $\underline{\mu}$ as follows: $\underline{\mu} = \sum_{i=1}^{\infty} k_i \delta_{\underline{x}^i}$. Thus, $\underline{\mu} \in \mathcal{A}$ and $\underline{\mu} \prec \mu$.

Proof of item $(A.4)$ of Theorem 4. Since $\mathsf{F}_{(p,q)}$ is monotone, we only need to prove that $\mathbb{E}\left(\tau_{\underline{\mu}}^{(p,q)}\right) = \infty$.

Using the definition of $\tau_{\underline{\mu}}^{(p,q)}$, we get

$$\tau_{\underline{\mu}}^{(p,q)} = \inf\{t \geq 0 : \delta_{\underline{x}^1}\mathsf{F}^t_{(p,q)}(1) = \delta_{\underline{x}^2}\mathsf{F}^t_{(p,q)}(1) = \ldots = 0\} = \inf\{t \geq 0 : \delta_{\underline{x}^1}\mathsf{F}^t_{(p,q)}(1) = 0\} = \tau_{\underline{x}^1}^{(p,q)}.$$

Using Lemma 5 we have, $\mathbb{E}\left(\tau_{\underline{x}^1}^{(p,q)}\right) = \mathbb{E}(H_1^X) = \infty$. Thus, $\mathbb{E}\left(\tau_{\mu}^{(p,q)}\right) = \infty$.

■

4 Proof of item $(B.4)$ of Theorem 4

Now, let us consider two sequences of independent random variables, $(L_t)_{t\in\mathbb{N}}$ and $(R_t)_{t\in\mathbb{N}}$, where for all $t \in \mathbb{N}$,

- The random variable L_t has the following distribution $\mathbb{P}(L_t = 0) = (1-\alpha)^N$ and $\mathbb{P}(L_t = k) = \alpha(1-\alpha)^{N-k}, \forall k \in \{-N, \ldots, -1\}$;
- R_t has the following distribution $\mathbb{P}(R_t = 0) = \beta$ and $\mathbb{P}(R_t = -1) = 1 - \beta$.

The process W^N is a Markov chain on $\mathbb{Z}$, with the following transition diagram (see Figure 3)

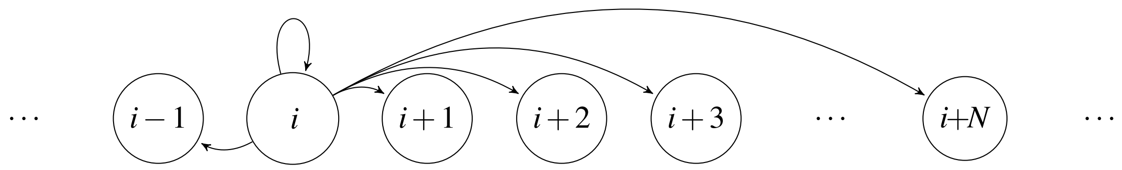

Figure 3: *Transition of process* W^N.

The $W^N = (W_t^N)_{t\in\mathbb{N}}$ is given by

$$W_t^N = W_0^N + \sum_{k=1}^{t}(R_k - L_k)$$

where $(R_t - L_t)_{t\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables.

For each $i \in \mathbb{Z}$

$$\begin{aligned}
\mathbb{P}(W_t^N = i+N | W_{t-1}^N = i) &= \mathbb{P}(R_t = 0)\mathbb{P}(-L_t = N) \\
\mathbb{P}(W_t^N = i-1 | W_{t-1}^N = i) &= \mathbb{P}(R_t = -1)\mathbb{P}(-L_t = 0) \\
\mathbb{P}(W_t^N = i | W_{t-1}^N = i) &= \mathbb{P}(R_t = 0)\mathbb{P}(-L_t = 0) + \mathbb{P}(R_t = -1)\mathbb{P}(-L_t = 1).
\end{aligned}$$

When $k \in \{1, \ldots, N-1\}$

$$\mathbb{P}(W_t^N = i+k | W_{t-1}^N = i) = \mathbb{P}(R_t = 0)\mathbb{P}(-L_t = k) + \mathbb{P}(R_t = -1)\mathbb{P}(-L_t = k+1).$$

Thus, its transition probabilities from state i to state j, p_{ij} is given by

$$p_{ij} = \begin{cases} \beta\alpha(1-\alpha)^{N-k} + (1-\beta)\alpha(1-\alpha)^{N-(k+1)} & \text{for} \quad j \in \{i+1, i+2, \ldots, i+N-1\} \\ & \text{and } k = j-i; \\ \beta(1-\alpha)^N + (1-\beta)\alpha(1-\alpha)^{N-1} & \text{for} \quad j = i \\ (1-\beta)(1-\alpha)^N, & \text{for} \quad j = i-1; \\ \beta\alpha, & \text{for} \quad j = i+N; \\ 0, & \text{at all the other cases.} \end{cases} \tag{6}$$

We get,

$$\mathbb{E}(R_t) = \beta - 1 \text{ and } \mathbb{E}(L_t) = \frac{1-(1-\alpha)^N}{\alpha} - 1 - N + (1-\alpha)^N.$$

Moreover, through the law of large numbers

$$W_t^N \to -\infty \text{ with probability 1 when } \beta < f_N(\alpha). \tag{7}$$

Given the processes W^N, we denote for $i > 0$

$$H_i^{W^N} = \inf\{t \geq 0 : W_t^N = 0 \text{ and } W_0^N = i\}.$$

From (7), we get that $\mathbb{E}(H_i^{W^N})$ is finite when $\beta < f_N(\alpha)$.

4.1 The operator E_N

Let us define the *extension* operator, $\mathsf{E}_N : \mathcal{A} \to \mathcal{A}$, which act in the following way.

First, we will define its action on δ-measures, δ_x, whose $x \in \Delta$ and $\mathsf{length}(x) > 1$. Clearly, there are positions $i_0 < j_0$ such that $x_{i_0} = x_{j_0} = 1$ and $x_k = 0$ for $k < i_0$ or $k > j_0$.

After applying E_N, each one among the components; $x_{i_0-N}, x_{i_0-N+1}, \ldots, x_{i_0-1}$; will be independent from each other: become 1 (one) with probability α or stay 0 (zero) with probability $1-\alpha$. On the other side, the component x_{j_0-1} and x_{j_0} becomes 1 (one) and 0 (zero), respectively with probability $(1-\beta)$ or x_{j_0-1} and x_{j_0} staying the same with probability β. Thus, we will get positions $i_1 < j_1$. Continuing with this argumentation, after applying E_N during t consecutive times, E_N^t, we will get positions i_t and j_t. The measures δ_x when $\mathsf{length}(x) \in \{0, 1\}$ are invariant distributions to E_N.

The E_N is a linear operator. Therefore, for $\mu \in \mathcal{A}$ we get $\sum\limits_{x^i \in \Delta} k_i(\delta_{x^i}\mathsf{E}_N) = \mu\mathsf{E}_N$.

For $x \in \Delta$ where $\mathsf{length}(x) = W_0^N + 1$, the length of the island on time t, $\delta_x \mathsf{E}_N^t$, decreases by one is given by the Markov chain W_t^N. Therefore, if W^N is negative with probability one when t goes to infinity, $\delta_x \mathsf{E}_N^t \to \delta_0$ when $t \to \infty$. In addition, the length of islands at $\delta_x \mathsf{E}_N^t$ is bigger than the length of islands at $\delta_x \mathsf{F}_{(0,N)}^t$. This relation among W^N, E_N and $\mathsf{F}_{(0,N)}$ will be usefull to prove the (B.4).

Lemma 6. *Given $x \in \Delta$ and δ_x its concentrated measure*

$$\delta_x \mathsf{E}_N(1) \geq \delta_x \mathsf{F}_{(0,N)}(1).$$

Proof. Direct from E_N definition.

■

Lemma 7. *Given μ, if*

$$\lim_{t\to\infty} \mu \mathsf{E}_N^t = \delta_0, \textit{ then } \lim_{t\to\infty} \mu \mathsf{F}_{(0,N)}^t = \delta_0.$$

Prova. Using the Lemma 6

$$\mu \mathsf{E}_N^t(1) = \sum_{x^i \in \Delta} k_i (\delta_{x^i} \mathsf{E}_N^t)(1) \geq \sum_{x^i \in \Delta} k_i (\delta_{x^i} \mathsf{F}_{(0,N)}^t)(1) = \mu \mathsf{F}_{(0,N)}^t(1).$$

■

Now, let us describe a relation between E_N and W^N. The action of E_N on the left-side of i_0 to obtain a new position i_1 is described by random variable L_1, once

$$\mathbb{P}(i_1 = i_0) = \mathbb{P}(x_{i_0-N} = \ldots = x_{i_0-1} = 0 \text{ and } x_{i_0} = 1) = (1-\alpha)^N = \mathbb{P}(L_1 = 0).$$
$$\mathbb{P}(i_1 = i_0 - 1) = \mathbb{P}(x_{i_0-N} = \ldots = x_{i_0-2} = 0 \text{ and } x_{i_0-1} = 1) = (1-\alpha)^{N-1}\alpha = \mathbb{P}(L_1 = 1).$$

Therefore, for $k \in \{1, \ldots, N\}$

$$\begin{aligned}\mathbb{P}(i_1 = i_0 - k) &= \mathbb{P}(x_{i_0-N}, \ldots, x_{i_0-(k-1)} \text{ stay zero and } x_k \text{ become one}) \\ &= (1-\alpha)^{N-k}\alpha \\ &= \mathbb{P}(L_1 = k).\end{aligned}$$

In a general way, for E_N^t

$$\mathbb{P}(i_t = i_{t-1} - k) = \mathbb{P}(L_t = k).$$

On the other side,

$$\mathbb{P}(j_t = j_{t-1}) = \mathbb{P}(x_{j_t} \text{ and } x_{j_{t-1}} \text{ stay the same}) = \mathbb{P}(x_{j_t} \text{ stay } 1) = \beta,$$

and

$$\mathbb{P}(j_t = j_{t-1} - 1) = \mathbb{P}(x_{j_{t-1}-1} = 1 \text{ and } x_{j_{t-1}} = 0) = (1-\beta).$$

So,

$$\mathbb{P}(j_t = j_{t-1}) = \mathbb{P}(R_t = 0) \text{ and } \mathbb{P}(j_t = j_{t-1} - 1) = \mathbb{P}(R_t = -1).$$

Lets $x \in \Delta$, its respective $\delta-$measure δ_x and $W_0^N = \mathsf{length}(x) - 1$. Let us see the evolution of $\delta_x \mathsf{E}_N$. The W_1^N indicates the length, minus one, of that island after E_N action. So, W_t^N indicates the length, minus one, of that island after t applications of E_N. If $(\delta_x \mathsf{E}_N^{t-1})\mathsf{E}_N = \delta_x \mathsf{E}_N^{t-1}$ then the respective island has length one. In this case, following the relation established between W^N and E_N, we have $W_t^N = 0$.

4.2 Another representation of $\mathsf{F}_{(p,q)}$

We can write the operator $\mathsf{F}_{(p,q)}$ in another way. In Figure 4 we have the action of operator $\mathsf{F}_{(p,q)}$ on the left side for given values of p and q. On the right- side, we have the action of the function T_p composed with $\mathsf{F}_{(0,q-p)}$, which is given by

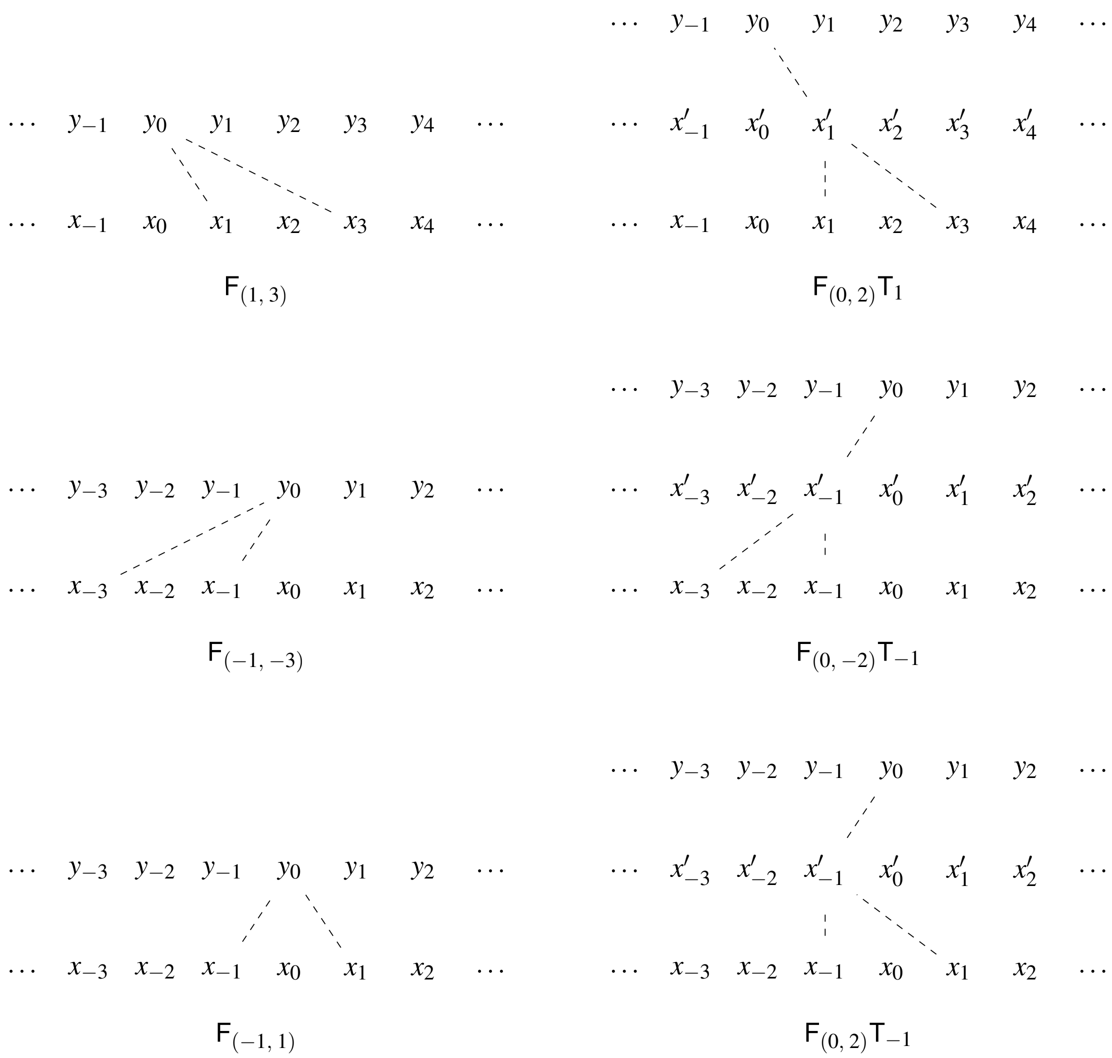

Figure 4: *On the left side, we schematized the interaction of the operators $\mathsf{F}_{(1,3)}$, $\mathsf{F}_{(-1,-3)}$, and $\mathsf{F}_{(-1,1)}$. The corresponding descriptions are placed on the right side through the composition with a translation operator. The time is running from the south to the north.*

$$\mathsf{F}_{(p,q)} = \mathsf{F}_{(0,q-p)}\mathsf{T}_p. \tag{8}$$

Lemma 8. *Let $x \in \Delta$, for each $t \in \mathbb{N}$*

$$\delta_x \mathsf{F}^t_{(p,q)}(1) = \delta_x \mathsf{F}^t_{(0,q-p)}(1).$$

Proof. Through (8), we have

$$\delta_x \mathsf{F}_{(p,q)} = \delta_x \left(\mathsf{F}_{(0,q-p)}\mathsf{T}_p \right).$$

Clearly T_p keeps the density of ones. So,

$$\delta_x \left(\mathsf{F}_{(0,q-p)}\mathsf{T}_p\right)(1) = \delta_x \mathsf{F}_{(0,q-p)}(1) \implies \delta_x \mathsf{F}_{(p,q)}(1) = \delta_x \mathsf{F}_{(0,q-p)}(1).$$

■

Lemma 9. *Given $x \in \Delta$ and δ_x its concentrated measure. If $\beta < f_N(\alpha)$ and $q - p = N$, then $\mathbb{E}(\tau_x^{(p,q)})$ is finite.*

Proof. Using (8)

$$\delta_x \mathsf{F}_{(p,q)} = \delta_x(\mathsf{F}_{(0,q-p)}\mathsf{T}_p).$$

So,

$$\delta_x \mathsf{F}_{(p,q)}(1) = \delta_x(\mathsf{F}_{(0,q-p)}\mathsf{T}_p)(1).$$

From the lemmas 6 and 7

$$\delta_x \mathsf{E}_N(1) \geq \delta_x \mathsf{F}_{(0,N)}(1); \tag{9}$$

and using

$$\tau_x^{\mathsf{E}_N} = \inf\{t \geq 0 : \delta_x \mathsf{E}_N^t = \delta_y \text{ and } \mathsf{length}(y) = 1\}$$

we get from (9) that

$$\mathbb{E}(\tau_x^{\mathsf{E}_N}) \geq \mathbb{E}(\tau_x^{(0,N)}) = \mathbb{E}(\tau_x^{(p,q)}).$$

Initially, we shall prove that $\mathbb{E}(\tau_x^{\mathsf{E}_N})$ is finite. Now, we used the relationship between E_N and W^N. As $\beta < f_N(\alpha)$ we have W_t^N equal to zero with probability one. Moreover, $\mathbb{E}(\tau_x^{\mathsf{E}_N}) = \mathbb{E}(H_{\mathsf{length}(x)}^{W^N})$, which is finite.

As a result, the length of the island is currently equal to one, with probability one. Then, using the relation between $\mathsf{F}_{(0,N)}$ acting on the island whose length is one with W^1, we can conclude: if $\beta < f_N(\alpha)$ then $\mathbb{E}(\tau_x^{(0,N)})$ is finite.

■

Proof of item $(B.4)$ of Theorem 4.

Lets $N = q - p$, $y \in \{x^1, x^2, \ldots\}$ and $\mathsf{length}(y) = \max\{\mathsf{length}(x^i); i = 1, 2, \ldots\}$. Using (8)

$$\begin{aligned}
\tau_\mu &= \inf\{t \geq 0 : \delta_{x^1}\mathsf{F}_{(p,q)}^t(1) = \delta_{x^2}\mathsf{F}_{(p,q)}^t(1) = \ldots = 0\} \\
&= \inf\{t \geq 0 : \delta_{x^1}\mathsf{F}_{(0,N)}^t(1) = \delta_{x^2}\mathsf{F}_{(0,N)}^t(1) = \ldots = 0\} \\
&= \tau_y,
\end{aligned}$$

By the Lemma 9, $\mathbb{E}(\tau_y)$ is finite. So, $\mathbb{E}(\tau_\mu)$ is finite.

■

5 Proof of Theorem 3

To prove Theorem 3, we shall use coupling[13, 11, 9] between the length o the island on time t, $\delta_x \mathsf{F}_{(p,q)^t}$ and the birth and death process X.

Let us take $p < 0 < q$ and $x \in \Delta$ with $\mathsf{length}(x) = 1$. In this case, the quantity of ones on the system is described by a birth-death process with the absorbing state, zero. Moreover, the distance between two consecutive ones is always the same, more precisely it is $N = q - p$. Given a natural value t, if $\delta_x \mathsf{F}^t_{(p,q)} \neq \delta_0$, then we shall get random positions (i_t, j_t), where with probability one $i_t < j_t$, $x_k = 0$ for all $k \leq i_t$ or $k \geq j_t$ and $x_{i_t+1} = x_{j_t-1} = 1$. If $\delta_x \mathsf{F}^t_{(p,q)} = \delta_0$, then there was moment t for which $j_{t-1} = i_{t-1} + 2$ and $j_t < i_t$.

The random variables i_t and j_t are related to the transition probabilities of the operator $\mathsf{F}_{(p,q)}$. It is defined by:

$$\mathbb{P}(i_t = i_{t-1} - q, j_t = j_{t-1} + p) = \begin{cases} 0, & \text{if} \quad j_t < i_t; \\ \theta(1|01)\theta(0|10), & \text{other cases.} \end{cases}$$

$$\mathbb{P}(i_t = i_{t-1} + q, j_t = j_{t-1} - p) = \begin{cases} 0, & \text{if} \quad j_t < i_t; \\ \theta(0|01)\theta(1|10), & \text{other cases.} \end{cases}$$

$$\mathbb{P}(i_t = i_{t-1} - q, j_t = j_{t-1} - p) = \begin{cases} 0, & \text{if} \quad j_t < i_t; \\ \theta(1|01)\theta(1|10), & \text{other cases.} \end{cases}$$

$$\mathbb{P}(i_t = i_{t-1} + q, j_t = j_{t-1} + p) = \begin{cases} 0, & \text{if} \quad j_t < i_t; \\ \theta(0|01)\theta(0|10), & \text{other cases.} \end{cases}$$

where $\theta(.|.)$ is on (2). Figure 5 illustrates the possible transitions of the process $\delta_x \mathsf{F}_{(-1,1)}$, where the initial measure is concentrated in an island whose length is one. In this case, we get $i_0 = -1$ and $j_0 = 1$. On the right side is presented the following possible transitions, values of i_1, j_1:

- Become (a), $i_1 = -2, \quad j_1 = 0$. It happens with probability $\theta(1|01)\theta(0|10)$;
- Become (b) $i_1 = 0, \quad j_1 = 2$. It happens with probability $\theta(0|01)\theta(1|10)$;
- Become (c) $i_1 = -2, \quad j_1 = 2$. It happens with probability $\theta(1|01)\theta(1|10)$;
- Become (d) $i_1 = 1, \quad j_1 = -1$. It happens with probability $\theta(0|01)\theta(0|10)$.

Lemma 10. *Given $p < 0 < q$, $0 < \alpha < 1$ and $\beta > 1/2\alpha$. If $\mathsf{length}(x) = 1$, then*

$$\lim_{t \to \infty} \delta_x F^t_{(p,q)} \neq \delta_0.$$

Proof. We will prove the following equivalent proposition: Given $0 < \alpha < 1$ and $\beta \geq 1/2\alpha$. If $\mathsf{length}(x) = 1$, then the probability of $i_t \to -\infty$ and $j_t \to \infty$ when $t \to \infty$ is positive.

We define a birth-death process with absorbing state zero, $X = (X_t)_{t>0}$, where for $a > 0$,

$$\begin{aligned} \mathsf{P}(X_{t+1} = a + 1 | X_t = a) &= \theta(1|01)\theta(1|10) \\ \mathsf{P}(X_{t+1} = a - 1 | X_t = a) &= \theta(0|01)\theta(0|10) + \theta(1|01)\theta(0|10) + \theta(0|01)\theta(1|10). \end{aligned}$$

The process X decrease, i.e., $\mathbb{P}(X_{t+1} < X_t) = 1$ for the random positions (i_t, j_t): both drift to the left or both drift to the right, or the i_t drifts to the right and j_t drift to the left, i.e., the (a) or (b) or (d) happens. And the process X increase, i.e., $\mathbb{P}(X_{t+1} > X_t) = 1$ when the i_t drift to the

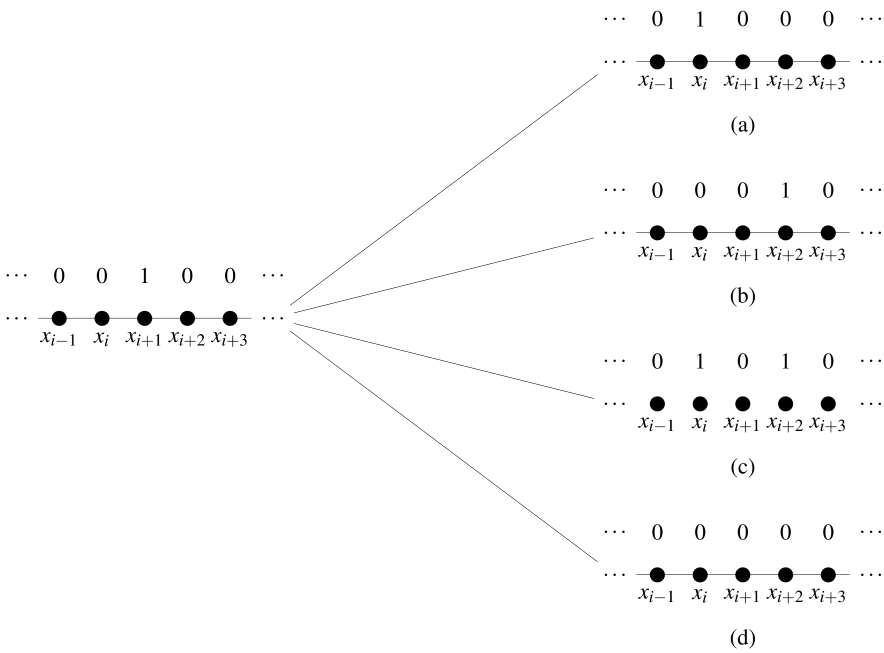

Figure 5: *Possible transitions of the* $\delta_x \mathsf{F}_{(-1,1)}$, *whose* $\mathsf{length}(x) = 1$.

left and j_t drift to the right, i.e., (c) happens. Since X is a birth and death process, sufficient conditions to be absorbed with a probability of less than one are known[17]. Using the relation between the process X and $\delta_x \mathsf{F}^t_{(p,q)}$ where $\mathsf{length}(x)$ is finite this sufficient condition happens if only if

$$\theta(1|01)\theta(1|10) > \theta(0|01)\theta(0|10) + \theta(1|01)\theta(0|10) + \theta(0|01)\theta(1|10).$$

Using (2) it means $\beta > 1/2\alpha$. Therefore, the quantity of times that (c) happens is bigger than the other ones with positive probability.

■

Proof of Theorem 3. Let us consider $\underline{\mu}$ and μ a convex combination of $\delta_{x_1}, \delta_{x_2}, \dots$ and $\delta_{\underline{x_1}}, \delta_{\underline{x_2}}, \dots$ respectively. As we know, each $\delta_{\underline{x_i}} \prec \delta_{x_i}$ where $\mathsf{length}(x_i) = 1$. By the Lemma 2

$$\delta_{\underline{x_i}} F^t_{(p,q)} \prec \delta_{x_i} F^t_{(p,q)}$$

and trough the Lemma 10

$$\lim_{t\to\infty} \delta_{\underline{x_i}} F^t_{(p,q)}(1) > 0 \Rightarrow \lim_{t\to\infty} \underline{\mu} F^t_{(p,q)}(1) > 0 \Rightarrow \lim_{t\to\infty} \mu F^t_{(p,q)}(1) > 0.$$

■

6 Numerical study

Some research areas have received more attention since building the first digital computer programmable. It is due to the possibility of performing a numerical simulation of some processes, previously developed by hand. A classic example of this is historic. Turing simulation(Turing

1952) studied a reaction-diffusion morphogen system[18]. Nowadays, the use of computers in the study of mathematical/physical models is widely used. Even to assist on the directions we should take on the theoretical studies. In this spirit, numerical methods are used in PCA studies, which are fruitful.

We will use mean-field approximation [4], MFA for simplicity. The MFA gives us a deterministic representation of the evolution of densities of zeros and ones. Usually, writing and analyzing the MFA is a first step in guessing the behavior of the system.

Let us take$(Y_i)_{i\in\mathbb{Z}}$ and $(Z_i)_{i\in\mathbb{Z}}$ two sequences of independent random variables. Moreover, for $i \in \mathbb{Z}$

$$\mathbb{P}(Y_i = 1) = \alpha \text{ and } \mathbb{P}(Y_i = 0) = 1-\alpha; \quad \mathbb{P}(Z_i = 1) = \beta \text{ and } \mathbb{P}(Z_i = 0) = 1-\beta.$$

Let x_i^t be the random variable that describes the state of the configuration x on the position i at time t. So, using the probability transition of $\mathsf{F}_{(p,q)}$ (see (2)) and that $(x_i^t)^2 = x_i^t$,

$$x_i^t = (x_{i+l}^t x_{i+r}^t)(1 - Y_i - Z_i) + x_{i+l}^t Z_i + x_{i+r}^t Y_i \tag{10}$$

Let us assume the initial measure, μ, uniform and define $\rho_i^t = \mathbb{E}_\mu(x_i^t)$. Since μ does not depend on i, we assume $\rho^t = \rho_i^t$. Taking expectation of both sides of (10) and assuming x_i^t and x_j^t independents each other. We obtain the following difference equation

$$\rho^{t+1} = \rho^t(\alpha+\beta) + (1-\alpha-\beta)(\rho^t)^2 \text{ where } (\alpha,\beta) \in (0,1)^2. \tag{11}$$

When we take $\rho^{t+1} = \rho^t$ in (11), we have a second degree equation to ρ^t. Moreover, their roots are zero and one, which agree with the actual process.

It is easy to conclude for $\rho^0 \in (0,1)$ the dichotomy to the sequence $\rho^0, \rho^1, \rho^2, \ldots$.

- $\rho^{t+1} > \rho^t$ if and only if $\beta > f_1(\alpha)$,
- $\rho^{t+1} < \rho^t$ if and only if $\beta < f_1(\alpha)$.

In both cases, $\rho^0, \rho^1, \rho^2, \ldots$ is a monotone sequence. Moreover, on the increasing case one is its upper limit. So, ρ^t goes to 1 when t tends to infinity. By similar argumentation, on the decreasing case ρ^t goes to 0 when t tends to infinity.

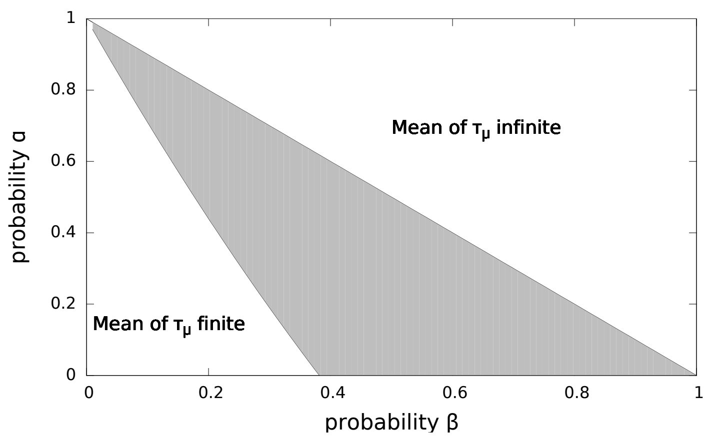

Figure 6: *Here, we illustrate the items $(A.4)$ and $(B.4)$ from Theorem 4. We take $N = 2$. We do not have any information for the gray region. The upper curve, $f_1(\alpha)$, is the transition line obtained by the Mean-field approximation.*

Qualitatively, the approximation reflects the existence of two behaviors when $\rho^0 \in (0,1)$. On the first one, ρ^t tends to 1 when t tends to infinity. On the second one, ρ^t tends to 0 when t tends to infinity. However, from the MFA if $\alpha > 1-\beta$ then $\rho^t \to 1$ when $t \to \infty$ for $\rho^0 \in (0,1)$; which do not agree with Theorem 1. Figure 6 illustrates the transition line, obtained by the MFA.

It is assumed that mean field models are essentially trivial. However, even the mean field models can exhibit surprising behavior[19]. As we are dealing with the influence of the neighborhood at PCA, maybe a mean field interaction[20] could be more appropriate to apply to our study.

Here, we will need to define a finite space with periodic boundary conditions, $\mathbb{Z}_n$ the set of remainders modulo n, where n is an arbitrary natural number. Let us consider the set of states $\Omega_n = \{0,1\}^n$. We call elements of Ω_n *periodic configurations.* The periodic configurations are finite sequences of zeros and ones, now we imagine these sequences to have a periodic form. For each $C \in \Omega_n$ de define $|C| = n$.

We are interested in the case when $p < 0 < q$. Thus, we performed the simulation of the process with $\mathsf{F}_{(-1,1)}$, where the system has $|C| = 200$, and the component of position 100 starts on state 1 and the others on state zero. On Figure 7 we illustrate the action of $\mathsf{F}_{(-1,1)}$ when (α,β) belongs to the set $\{(1,1); (0.8,0.8); (1,0.3); (0.3,1)\}$. The case $\alpha = \beta = 1$ is the deterministic evolution of the PCA.

The computer simulations suggest that when $p < 0 < q$ and for α and β sufficiently close to one, the process converges to an invariant measure which is not a convex combination of the measures concentrated on the configuration, with all the components are at the same state. Still in this direction, what is the speed at which the process will converge to this measure? Another question is: will the ones always "spread out" throughout the system?

7 Open problems

We conclude with several open problems.

At this work, we have explored the effect of the neighbor's position on the dynamics of a class of one-dimensional PCA. The main results confirm that its relevance. According we know, this is one of the first formal results in this direction. It brings a set of questions, we quote some of them.

Problem 1. *To the class of PCA that we have studied, what happens if we take a uniform initial distribution?*

Theorem 4 establishes two possible regimes(described by (A.4) and (B.4)). So, we get a region for which we do not have any knowledge for a given N value.

Problem 2. *Is there another intermediate regime instead of that present by (A.4) and (B.4) ?*

Theorem 2 and item (B.4) consider $\beta < f_N(\alpha)$. Clearly, for a fixed N this region is non empty. Moreover, fixed α and given $N < M$ we get $f_N(\alpha) > f_M(\alpha)$. So, the region $\beta < f_N(\alpha)$ decreases when N increases.

Problem 3. *Is it possible to obtain qualitative compatible versions of Theorem 2 and item (B.4) not dependent on N?*

The impact of neighbors at PCA on dimensions bigger than one needs attention.

Problem 4. *Is it possible to built a PCA with dimensions greater than one in which dynamics are driven by the location of the neighbor?*

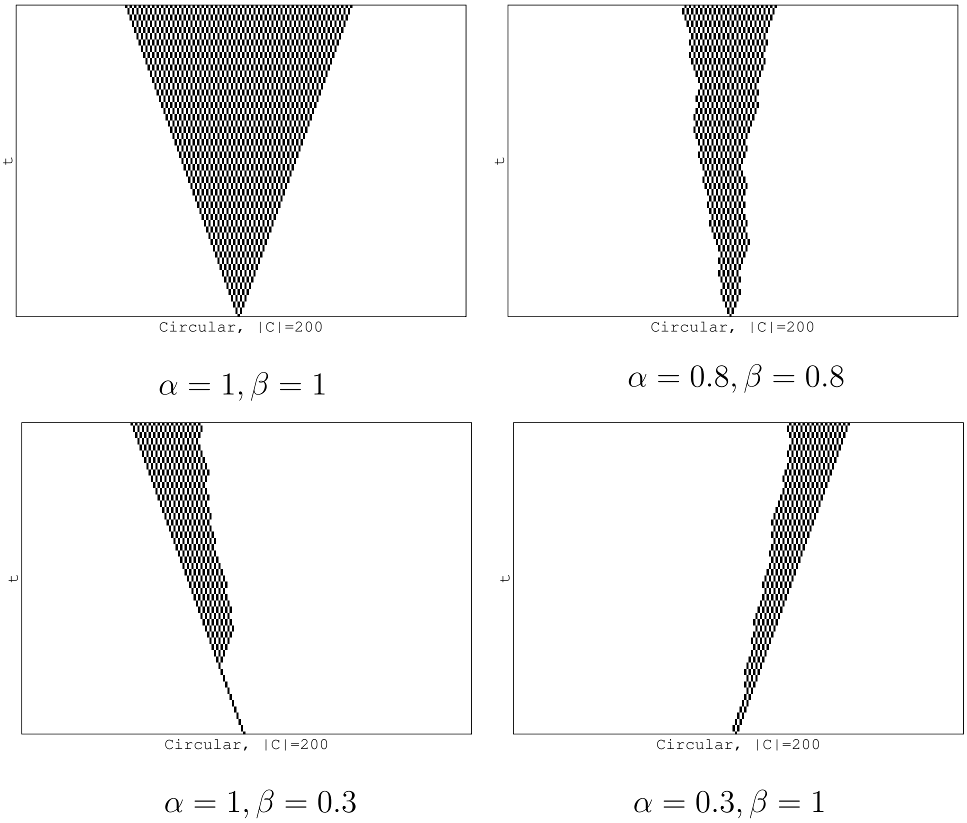

Figure 7: Four space-time diagrams for $\mathsf{F}_{(-1,1)}$. In each Figure, we took $|C| = 200$ and $t = 0, \ldots, 50$.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and suggestions, which certainly improved the presentation and quality of the paper. C. D. B. da Silva cordially thanks the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES, which supported this work.

Data Availability

Data sharing not applicable-no new data generated. Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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